MA 410

Multivariable Calculus

Amsha Krishan (211-G)

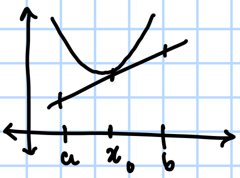
Midterm - 30%.
Endterm - 40%.
Quizes - 30% - Each week
(HWs + Quizzes)
First Quiz - Next thursday
Moodle - Book
Quizes - From Homework
Book 1 - Spivak
Book 2 - Munkres

6^{th} Jan:

Recall: Single-variable calculus:

differentiation

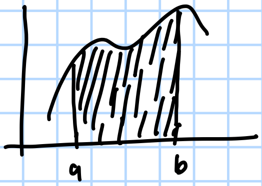
$f:(a,b) \longrightarrow \mathbb{R}$



Differentiation → geometrically it is the slope of the tangent line

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Integration:



$f:(a,b) \longrightarrow \mathbb{R}$

partition of $[a,b]$ by $\mathcal{P}$

upper sums and lower sums

$U(P,f)$

$L(P,f)$

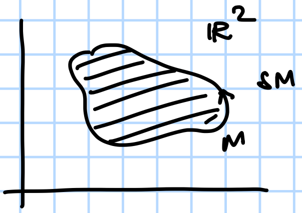
f is riemman integrable if upper and lower sum converge

$\forall \varepsilon > 0, \exists P \in \mathcal{P}$ st $|U(P,f) - L(P,f)| < \varepsilon$

$\Rightarrow \int_a^b f(x)dx = \sup_{\mathcal{P}} L(P,f) = \inf_{\mathcal{P}} U(P,f)$

FTC: $\int_a^b F' = F(b) - F(a)$

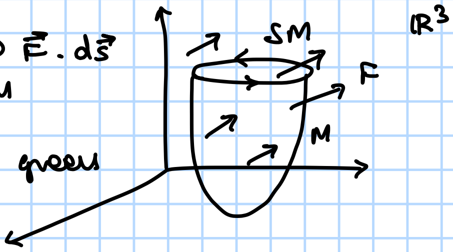
Green theorm:



$$\iint_M \left(\frac{\delta \beta}{\delta x} - \frac{\delta \alpha}{\delta y}\right) dA = \int_{\delta M} \alpha dx + \beta dy$$

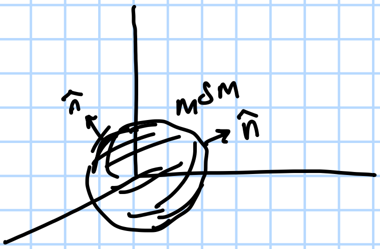
Stokes theorm:

$$\iint_M (\underbrace{\vec{\nabla} \times \vec{F}}_{\text{curl of } F})\, dA = \oint_{dM} \vec{F} \cdot d\vec{s}$$



strokes theorm is a genralisation of green theorm

Divergence theorm:



$$\iiint_M \underset{(\vec{\nabla} \cdot \vec{F})}{\operatorname{div} \vec{F}} \cdot dv = \iint_{\delta M} \vec{F} \cdot \hat{n} \cdot dA$$

Note: FTC means $\delta([a,b]) = \{a,b\}$

$$\int_a^b F' = F(b) - F(a)$$

when got to 2/3 dimentionales, give us greens theorm / stokes theorm / div theorm

we need a general case of FTC and these higher dim theorms for any variable, it is called: **stoke's theorm**:

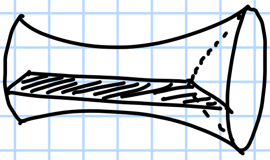
$$\int_M d\omega = \int_{\delta M} \omega$$

← differential form (ω)

← Boundary of manifold (δM)

← n-dimmentional manifold (M)

Note: This stokes theorm is the goal of the course.



↙ Many ways to integrate this

Note : New form of derivative

$$0 = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0) - f'(x_0).h}{h}$$

This
$g : (-\varepsilon, \varepsilon) \to \mathbb{R}$
$h \mapsto f(x_0+h) - f(x_0)$
$T : (-\varepsilon, \varepsilon) \to \mathbb{R}$
$h \mapsto f'(x_0).h$

T is a good aproximation to g, for small values of h.

Note : T is a linear approximation to g.

Eg :
$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

say f is differentiable at x_0 if $\exists\, T : \mathbb{R}^n \to \mathbb{R}^m$ linear transformations
s.t $\lim_{h \to 0} \frac{|f(x_0+h) - f(x_0) - T(h)|}{h} = 0$

↖ we use norm as this is a vector

If this happen, we say $Df(x_0) = T$

Note : Derivative is a linear transformation, and in calculus we want to transform funtions and approximate them with linear funtions.

$T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if $T(au+bv) = aT(u) + bT(v)$, $\forall a, b \in \mathbb{R}$, $\forall u, v \in \mathbb{R}^n$

Tommorow : Review of linear algebra, vector spaces, linear transformation, rank of a matrix, determinant, norms and inner products

Home work 1 : Testing on linear algebra basics

7th Jan:

Email: anushamk@math.iitb.ac.in

textbook: Spivak / Calculus on manifolds

textbook: Munkres / Analysis on manifolds

Review of linear algebra:

I. Vector space over $\mathbb{R}$:

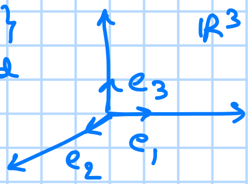
$V, +, \cdot$ satisfying:
① $u+v \in V, \forall u, v \in V$
② $a.v \in V, \forall v \in V, a \in \mathbb{R}$
③ commutivity $u+v=v+u$
④ assosiativity $(u+v)+w = u+(v+w)$

Basis(β):

- $\beta = \{v_1, \dots v_n\}$ is said to be a basis for V if β spans V & $v_1, \dots v_n$ are lin ind.
- Say $|\beta| = n$ is the dimention of V

Example: $V = \mathbb{R}^n$ is a vectorspace of dim n

$\beta = \{e_1, \dots \dots e_n\}$ is called standard basis for $\mathbb{R}^n$



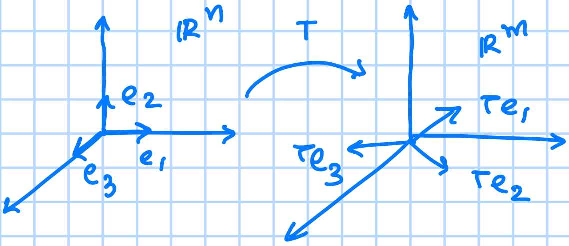
$e_1 = (1, 0, \dots \dots 0)$
$e_2 = (0, 1, \dots \dots 0)$
$\vdots$
$e_n = (0, 0, \dots \dots 1)$

II linear transformation:

Say $T: V \longrightarrow W$ is a linear transformation if

$$T(au + bv) = aT(u) + bT(v)$$
$$\forall a, b \in \mathbb{R}$$
$$\forall u, v \in V$$

Example:



where basis go is enough to determine T.

For $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ if $T(e_i) = b_i = \begin{bmatrix} b_i^1 \\ \vdots \\ b_i^m \end{bmatrix}$, matrix rep of this is:

$m \times n$ matrix: $\begin{bmatrix} T(e_1) & \cdots & T(e_n) \end{bmatrix}_{m \times n}$ = M is called the matrix of linear transformation of T.

III. Norm and inner product:

Denote vectors in $\mathbb{R}^n$ by $x = (x^1, \dots x^n)$

Euclidian norm is defined by:

$$|x| = \sqrt{(x^1)^2 + (x^2)^2 + \cdots (x^n)^2} = \sqrt{\sum_{i=1}^{n} (x^i)^2}$$

Theorm : If $x, y \in \mathbb{R}^n$, $a \in \mathbb{R}$, than the following are true :

① $|x| \geq 0$, and $|x| = 0$ iff $x = 0$
② $|\Sigma x^i y^i| \leq |x||y|$, equality holds if x and y are linearly dep.
③ $|x + y| \leq |x| + |y|$ (triangle inequality)
④ $|a.x| = |a|.|x|$

proof : ← done

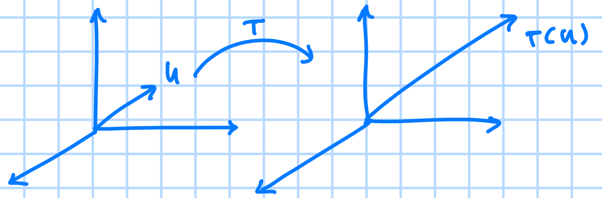
More generally, if $\|\cdot\|: V \to \mathbb{R}$ satify ①, ③, and ④ we say $\|\cdot\|$ is a norm on V.

↙ l^1 norm is $\|x\| = \Sigma |x^i|$

Note : an l^1 norm in the inf dim is an example not satifying ② ← see

sup norm on $\mathbb{R}^n$ is an exaple satisfying ①,③,④ and not ② ← see ↙ $\|f\| = \sup_{x \in K} |f(x)|$

Ex : If $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, show that there is a number M s.t $|T(h)| \leq M|h|$, $\forall h \in \mathbb{R}^m$



$$T = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} h^1 \\ \vdots \\ h^m \end{bmatrix} = \begin{bmatrix} \sum^m a_{1j} h^j \\ \vdots \\ \sum^m a_{nj} h^j \end{bmatrix}$$

ans : $T = [a_{ij}]_{n \times m}$ $h = (h^1, \ldots h^m)$

$$T(h) = \left(\sum_{j=1}^m a_{1j} h^j, \sum_{j=1}^m a_{2j} h^j, \ldots \sum_{j=1}^m a_{nj} h^j\right) \leftarrow$$

$$|T(h)| = \sqrt{\sum_{i=1}^n \left(\sum_{j=1}^m a_{ij} h^j\right)^2}$$

now with cauchy swarts,

$$|T(h)|^2 = \sum_{i=1}^n \left(\sum_{j=1}^m a_{ij} h^j\right)^2 \leq \sum_{i=1}^n \left(\sum_{j=1}^m a_{ij}^2\right) |h|^2$$

$$= |h|^2 \left(\sum_{i=1}^n \sum_{j=1}^m a_{ij}^2\right) \leftarrow \text{This is } M^2$$

$$= |h|^2 M^2$$

Euclidian inner product : $\langle x, y \rangle = \sum_{i=1}^n x^i y^i$

Here $x = (x^1, \ldots x^n) \in \mathbb{R}^n$
$y = (x^1, \ldots x^n) \in \mathbb{R}^n$

Theorm : Let $x, y, x_1, y_1, x_2, y_2 \in \mathbb{R}^n$, $a \in \mathbb{R}$ then

① $\langle x, y \rangle = \langle y, x \rangle$ (symmetric)
② $\langle ax, y \rangle = \langle x, ay \rangle = a\langle x, y \rangle$
$\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$ } (Bilinear)
$\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$
③ $\langle x, x \rangle \geq 0$, $\langle x, x \rangle = 0$ iff $x = 0$ (position definatness)
④ $|x| = \sqrt{\langle x, x \rangle}$
⑤ $\langle x, y \rangle = \dfrac{|x+y|^2 - |x-y|^2}{4}$ (polorisation identity)

proof : $\langle x, y \rangle = \Sigma x^i y^i$

① $\langle x, y \rangle = \Sigma x^i y^i$
$\langle y, x \rangle = \Sigma y^i x^i = \Sigma x^i y^i = \langle x, y \rangle$

② $\langle ax, y \rangle = \Sigma a x^i y^i = \Sigma x^i (ay^i) = \langle x, ay \rangle$
$= \Sigma x^i (ay^i)$
$= a \Sigma x^i y^i$
$= a \langle x, y \rangle$

$$\langle x_1 + x_2, y\rangle = \sum (x_1^i + x_2^i)(y^i)$$
$$= \sum x_1^i y^i + \sum x_2^i y^i$$
$$= \langle x_1, y\rangle + \langle x_2, y\rangle$$

$$\langle x, y_1 + y_2\rangle = \sum (x^i)(y_1^i + y_2^i)$$
$$= \sum (x^i)(y_1^i) + \sum (x^i)(y_2^i)$$
$$= \langle x, y_1\rangle + \langle x, y_2\rangle$$

③ $\langle x, x\rangle = \sum (x^i)^2$ as $(x^i)^2 \geq 0 \ \forall\, i \in \{1, 2 \dots n\}$

so $\sum (x^i)^2 \geq 0$

now if $\langle x, x\rangle = 0$ then $\sum (x^i)^2 = 0$

wlog $(x^1)^2 = -\sum_{i=2}^{n} (x^i)^2$

$\geq 0 \quad \leq 0$

$\Rightarrow (x^1)^2 = 0$

$\Rightarrow x^1 = 0$

similarly $x^1 = 0$

$\Rightarrow x = 0$

now if $x = 0$ then $\langle x, x\rangle = 0$ is trivial.

④ $|x| = \sqrt{\langle x, x\rangle}$

now, $|x| = \sqrt{\sum (x^i)^2} = \sqrt{\sum (x^i)(x^i)}$
$= \sqrt{\langle x, x\rangle}$

⑤ $|x - y|^2 = \langle x - y, x - y\rangle$
$= \langle x, x - y\rangle - \langle y, x - y\rangle$
$= \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle$

$|x + y|^2 = \langle x + y, x + y\rangle$
$= \langle x, x + y\rangle + \langle y, x + y\rangle$
$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$

now $|x + y|^2 - |x - y|^2 = 4\langle x, y\rangle$

$\Rightarrow \dfrac{|x + y|^2 - |x - y|^2}{4} = \langle x, y\rangle$

Note : more generally if $\langle\langle \cdot, \cdot \rangle\rangle$ satisfies ①, ② and ③, we say $\langle\langle \cdot, \cdot \rangle\rangle$ is an inner product.

Dual space : let $(\mathbb{R}^n)^*$ denote the dual of vector space $\mathbb{R}^n$

$(\mathbb{R}^n)^* = \{ L : \mathbb{R}^n \to \mathbb{R}$ s.t L is a linear transformation $\}$

linear functional

Note : Dual space is also a vector space.

For $x \in \mathbb{R}^n$, define $\varphi_x \in (\mathbb{R}^n)^*$ by $\varphi_x(y) = \langle x, y\rangle$

Define $T : \mathbb{R}^n \longrightarrow (\mathbb{R}^n)^*$ by
$x \longmapsto \varphi_x$

Claim: T is a $\underbrace{\text{1-1}}_{\text{①}}$ $\underbrace{\text{linear transformation}}_{\text{②}}$

proof: ② $T(a.x) = Qax$
$= a\,Qx$
as $Qax(y) = \langle ax, y\rangle = a\langle x, y\rangle$ —— ①
$= a\,Qx(y)$

$T(x_1 + x_2) = Qx_1 + x_2$
$= Qx_1 + Qx_2$
as
$Qx_1+x_2(y) = \langle x_1 + x_2, y\rangle$
$= \langle x_1, y\rangle + \langle x_2, y\rangle$ —— ②
$= Qx_1(y) + Qx_2(y)$

from ①, ②, T is a linear transformation

①: Suppose $T(x) = 0$ then
$Qx = 0 = \langle x, y\rangle \quad \forall y \in \mathbb{R}^n$
$\Rightarrow \langle x, x\rangle = 0$
for $y = x$
$\Rightarrow x = 0$

$\therefore \forall\, T(x) = 0$
$\Rightarrow x = 0$
or Null $T = \{0\}$
$\Rightarrow$ T is 1-1

Given an $m \times n$ matrix $A = [a_{ij}]$
1) Column space of A = linear span of the columns of $A \subseteq \mathbb{R}^m$
its dim is called the column rank of A

2) Row space of A = linear span of the rows of $A \subseteq \mathbb{R}^n$
its dim is called the row rank of A

Theorm: column rank(A) = row rank(A) = r and we define rank(A) = r

Say an $n \times n$ matrix A is invertible if $\exists$ an $n \times n$ B s.t

$$AB = BA = I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

Theorm: $n \times n$ matrix A is invertible iff rank(A) = n
$n \times n$ matrix A is invertible iff $\det A \neq 0$

III Determinant :

$\leftarrow$ set of $n \times n$ matrix with real entries.

Define $\det(\cdot) : M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$

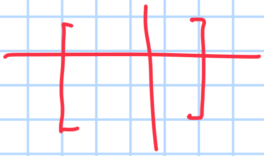
1) If $A \xrightarrow[\text{2 colums}]{\text{interchange}} B$ then

$\det B = -\det A$

2) det is linear in each column

3) $\det I_n = 1$

properties: 1) $\det(A.B) = \det(A).\det(B)$
2) $\det(A^T) = \det(A)$
3) $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$
where $A_{ij} = (i,j) \leftarrow$ minor of A



Theorm : If $x, y \in \mathbb{R}^n$, $a \in \mathbb{R}$, then the following are true:

① $|x| \geq 0$, and $|x| = 0$ iff $x = 0$

② $|\Sigma x^i y^i| \leq |x| |y|$, equality holds if x and y are linearly dep.

③ $|x + y| \leq |x| + |y|$ (triangle inequality)

④ $|a.x| = |a|.|x|$

proof : ① $|x| = \sqrt{\Sigma (x^i)^2}$ as

$(x^i)^2 \geq 0 \ \forall \ i \in \{1, 2 \ldots n\}$

$\Rightarrow \Sigma (x^i)^2 \geq 0$

$\Rightarrow \sqrt{\Sigma (x^i)^2} \geq 0$

$\Rightarrow |x| \geq 0$

now if $|x| = 0$ then

$\Sigma (x^i)^2 = 0$

$\Rightarrow (x^j)^2 = -(x^1)^2 - \cdots (x^{j-1})^2 - (x^{j+1})^2 \ldots - (x^n)^2$

$\Rightarrow (x^j)^2 \leq 0$ & $(x^j)^2 \geq 0$

$\Rightarrow (x^j)^2 = 0$

$\therefore x = 0 \quad \forall j$

so $|x| = 0 \Rightarrow x = 0$

also $x = 0 \Rightarrow |x| = 0$ is trivial

② $|\Sigma x^i y^i| \leq |x| \cdot |y|$ is called the cauly swarts inequality

here if x ad y are lin ind then

$\lambda x + y = 0$ for some $\lambda \in \mathbb{R}$

ad so

$|\Sigma x^i y^i| = |\Sigma (-\lambda x^i)(x^i)|$

$= |\lambda| (\Sigma (x^i)^2)$

$= |\lambda| |x| |x|$

$= |x| |y|$ (④ is used)

now, if $\lambda x + y \neq 0 \ \forall \ \lambda \in \mathbb{R}$ then

$|\lambda x + y| > 0$

$\Rightarrow \Sigma (\lambda x^i + y^i)^2 > 0$

$\Sigma (\lambda x^i + y^i)^2 > 0$

$\Rightarrow \lambda^2 \Sigma (x^i)^2 + \Sigma (y^i)^2 + 2\lambda \Sigma x^i y^i > 0$

as $\forall \lambda \Rightarrow \Delta < 0$

or $b^2 - 4ac < 0$

$\Rightarrow (\cancel{2} \Sigma x^i y^i)^2 - \cancel{4} (\Sigma (x^i)^2)(\Sigma (y^i)^2) < 0$

$\Rightarrow (\Sigma x^i y^i)^2 < \Sigma (x^i)^2 \, \Sigma (y^i)^2$

$\Rightarrow |\Sigma x^i y^i| < |x| \cdot |y|$

so, $|\Sigma x^i y^i| \leq |x| \cdot |y|$

③ $|x + y| \leq |x| + |y|$

as

$|x + y|^2 = \Sigma (x^i + y^i)^2$

$= \Sigma (x^i)^2 + \Sigma (y^i)^2 + 2 \Sigma (x^i y^i)$

$\leq \Sigma (x^i)^2 + \Sigma (y^i)^2 + 2 |x| \cdot |y|$

$= |x|^2 + 2 |x| \cdot |y| + |y|^2$

$= (|x| + |y|)^2$

$\Rightarrow |x + y| \leq |x| + |y|$

④ $|a \cdot x| = |a| \cdot |x|$

here $|a \cdot x| = |(ax^1, ax^2, \ldots ax^n)|$

$= \sqrt{a^2 (\Sigma (x^i)^2)}$

$= |a| \sqrt{\Sigma (x^i)^2}$

$= |a| \cdot |x|$

9th Jan :

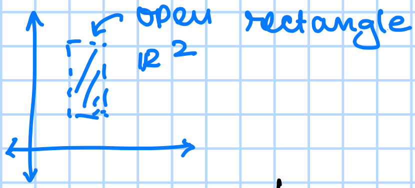
Quiz on Thursday 16th Jan

Review of topology of $\mathbb{R}^n$:

I. Subsets of $\mathbb{R}^n$:

In $\mathbb{R}^1$: (a,b) is open interval
$[a,b]$ is closed interval

In $\mathbb{R}^n$: closed rectangle: $[a_1,b_1]\times \cdots \times[a_n,b_n]$
open rectangle: $(a_1,b_1)\times\cdots(a_n,b_n)$



we say that $U\subset\mathbb{R}^n$ is open if $\forall x\in U$, $\exists$ an open rectangle A with $x\in A\subseteq U$

we say that $B\subseteq\mathbb{R}^n$ is closed in $\mathbb{R}^n\setminus B$ is open.

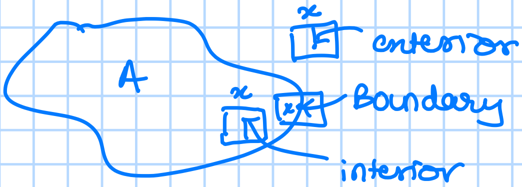
Note: In fact, U is open iff

$\forall x\in U, \exists\, \delta>0$ s.t $B_\delta(x)\subseteq U$ ← This is equivalent defination to using open rectangle.

Def: The interior of A is defined as $\{x\in\mathbb{R}^n \mid \exists$ an open rectangle $B\subseteq A$ s.t $x\in B\}$

Def: The exterior of A is defined as $\{x\in\mathbb{R}^n \mid \exists$ an open rectangle B s.t $x\in B$ and $B\subseteq\mathbb{R}^n\setminus A\}$

Def: The boundary of A is defined as $\{x\in\mathbb{R}^n \mid$ each open rectangle $x\in B$, satisfies $B\cap A\neq\emptyset$ and $B\cap(\mathbb{R}^n\setminus A)\neq\emptyset\}$



Def: A collection $\mathcal{O}$ of open sets is called an open cover for A if $A\subseteq\bigcup_{U\in\mathcal{O}} U$

Def: we say that A is compact if every open cover $\mathcal{O}$ for A has a finite subcover of sets which cover A.

Eg: $\left\{\left(\frac{1}{n},1\right)\middle|\, n\in\mathbb{N}\right\}=\mathcal{O}$ then $(0,1)\subseteq\bigcup_{n\geq 2}\left(\frac{1}{n},1\right)$ but

no finite subcover covers $(0,1)$

$\left\{\left(\frac{1}{n_1},1\right),\dots\left(\frac{1}{n_k},1\right)\right\}$ [order if finite subcover → contradiction] eg : $\mathbb{R}$ is not compact

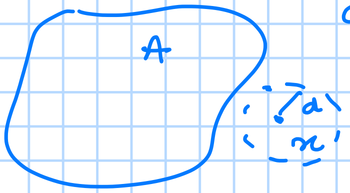
$\mathcal{O} = \{(-n, n) \mid n \geq 1\}$ covers $\mathbb{R}$ but no finite subcollection covers $\mathbb{R}$.

Theorm: (Hien-Borel) A set $A \subseteq \mathbb{R}^n$ is compact iff A is closed and bounded.

Exe: (Spivak 1-21)

(a) If A is closed and $x \notin A$, prove that $\exists d > 0$ s.t $|y - x| \geq d \ \forall y \in A$

Ans :



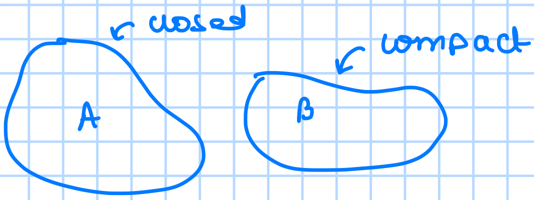
as $x \notin A \Rightarrow x \in \mathbb{R}^n \setminus A$
and as A is closed
$\Rightarrow \mathbb{R}^n \setminus A$ is open

so $\exists d > 0$ s.t
$B_d(x) \subseteq \mathbb{R}^n \setminus A$

$\Rightarrow \forall y \in A, |y - x| \geq d$

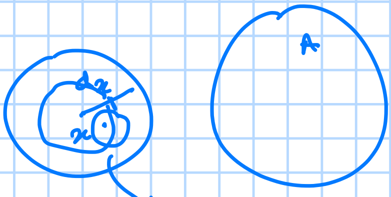
(b) If A is closed and B is compact and $A \cap B = \phi$ then show that $\exists d > 0$ s.t $|y - x| \geq d \ \forall \ y \in A$ and $x \in B$

Ans :



by (a) for each point $x \in B$, $\exists d_x > 0$ s.t
$B_{d_x}(x) \subseteq \mathbb{R}^n \setminus A$

then $\forall z \in B_{\frac{d_x}{2}}(x)$, $B_{\frac{d_x}{2}}(z) \subseteq B_{d_x}(x) \subseteq \mathbb{R}^n \setminus A$



$B_{\frac{dx}{2}}(z) \subseteq B_{dx}(x) \subseteq \mathbb{R}^n \setminus A$

By triagle inequality $\forall y \in B_{\frac{dx}{2}}(z)$
s.t $|y - z| < \frac{dx}{2}$ then

$|y - x| < |y - z| + |z - x|$
$< \frac{dx}{2} + \frac{dx}{2}$

$\Rightarrow |y - x| < dx$
so $y \in B_{dx}(x)$

$\therefore \forall y \in B_{\frac{dx}{2}}(z) \Rightarrow y \in B_{dx}(x)$

$\Rightarrow B_{\frac{dx}{2}}(z) \subseteq B_{dx}(x)$

now, $\forall z \in B_{\frac{dx}{2}}(x)$, $|w - z| \geq \frac{dx}{2} \ \forall \ w \in A$
$\subseteq \mathbb{R}^n \setminus A$

now, $\mathcal{O} := \{B_{\frac{dx}{2}}(x) \mid x \in B\}$

$\mathcal{O}$ is an open cover for B, and B is compact
so $\exists$ a finite subcover

$\{B_{\frac{dx_1}{2}}(x_1), \ldots B_{\frac{x_k}{2}}(x_k)\}$ covers B

Then $\forall x \in B$, $\exists i$ s.t

$$x \in B_{\frac{dx_i}{2}}(x_i)$$

for any $w \in A$

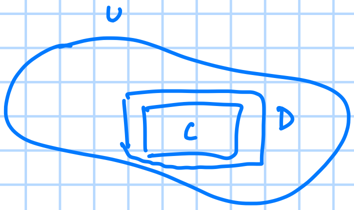
$$|x-w| \geq \frac{dx_i}{2} \geq d = \min\left\{\frac{dx_1}{2}, \ldots \frac{dx_k}{2}\right\}$$

so, $\forall x \in B$, $\forall w \in A$

$$|x-w| \geq d$$

Exe: (Spinker 1-22)
If U is open and $C \subset U$ is compact, show that $\exists$ compact set D s.t $C \subset$ interior D and $D \subset U$

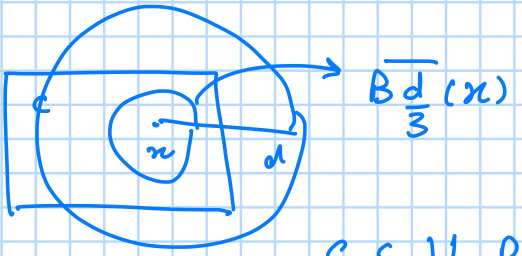
Ans:



C is compact, $A = \mathbb{R}^n \setminus U$
$\Rightarrow A$ is closed
and by the previous exersise
$\exists\, d > 0$ s.t
$|y-x| \geq d \quad \forall y \in A, x \in C$

$$\forall x \in C, \quad \overline{B_{\frac{d}{3}}(x)} \subseteq B_{\frac{d}{2}}(x)$$

and $\bar{B}_{\frac{d}{3}}(x) \cap A = \phi$



$$C \subseteq \bigcup_{x \in C} B_{\frac{dx}{3}}(x)$$

as C is compact, we cover C by finitely many of the

$$\left\{B_{\frac{d}{3}}(x_1), \ldots B_{\frac{d}{3}}(x_k)\right\}$$

$$C \subseteq \underbrace{\bigcup_{i=1}^{k} B_{\frac{d}{3}}(x_i)}_{\text{open}} \subseteq \underbrace{\bigcup_{i=1}^{k} \bar{B}_{\frac{d}{3}}(x_i)}_{\text{closed}} = D \leftarrow$$ as D is closed and Bounded, D is compact.

$$C \subseteq \underset{\text{open}}{V} \subseteq \underset{\text{compact}}{D} \quad \text{and} \quad D \cap A = \phi$$

III functions and continuity:

$f:\mathbb{R}^n \to \mathbb{R}^m$ is written as

$$f(x) = \left(f^1(x), \ldots \, f^m(x)\right)$$

where $f^i: \mathbb{R}^n \to \mathbb{R}$

Def: $\pi: \mathbb{R}^n \to \mathbb{R}^n$ denotes the identity function
$x \mapsto x$

Def: $\pi^i: \mathbb{R}^n \to \mathbb{R}$ is the ith projection function
$(x^1, x^2, \ldots x^i \ldots x^n) \mapsto x^i$

$\lim\limits_{x \to a} f(x) = b$ means that $\forall \varepsilon > 0, \exists \delta > 0$ s.t
$|f(x) - b| < \varepsilon$ when $|x - a| < \delta$

say that f is continuous at a if $\lim\limits_{x \to a} f(x) = f(a)$

f is cont iff $f^{-1}(U)$ is open when U is open.

f is cont iff $f^{-1}(B)$ is closed when B is closed.

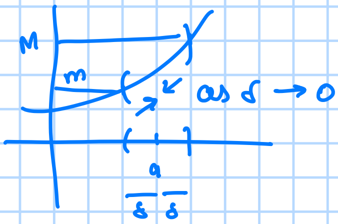
Theorm: If $f: A \to \mathbb{R}^n$ is cont and A is compact, then $f(A)$ is compact

Osilation of a function at a point a:

suppose $f: A \to \mathbb{R}$ is bounded then

$$M(a,f,\delta) = \sup\{f(x) \mid x \in A \text{ and } |x-a| < \delta\}$$
$$m(a,f,\delta) = \inf\{f(x) \mid x \in A \text{ and } |x-a| < \delta\}$$

Def: (osilation) $o(f,a) = \lim\limits_{\delta \to 0} M(a,f,\delta) - m(a,f,\delta)$



if f is cont
then $O(f,a) \longrightarrow 0$

13th Jan:

Recap: Osilation of a function at a point

$f : \underset{\subseteq \mathbb{R}^n}{A} \to \mathbb{R}$ is bounded

define for $\delta > 0$

$$M(a,f,\delta) = \sup\{f(x) \mid |x-a| < \delta\}$$
$$m(a,f,\delta) = \inf\{f(x) \mid |x-a| < \delta\}$$

osilation of f at a:

$$o(f,a) = \lim_{\delta \to 0} M(a,f,\delta) - m(a,f,\delta)$$

Theorm: let $f : A \to \mathbb{R}$ be bounded, $a \in A$, then f is cont at a iff $o(f,a) = 0$

proof:

($\Rightarrow$) suppose f is cont at a, given $\varepsilon > 0$, $\exists \delta > 0$ s.t

$$|f(x) - f(a)| < \varepsilon \quad \text{whenever} \quad |x-a| < \delta$$

So, $|x-a| < \delta \Rightarrow f(a) - \varepsilon < f(x) < f(a) + \varepsilon$

now, $M(a,f,\delta) \leq f(a) + \varepsilon$

$m(a,f,\delta) \geq f(a) - \varepsilon$

$\Rightarrow M(a,f,\delta) - m(a,f,\delta) \leq 2\varepsilon$

$\Rightarrow \lim_{\delta \to 0} M(a,f,\delta) - m(a,f,\delta) = 0 = o(f,a)$

($\Leftarrow$) suppose $o(f,a) = 0$, i.e given $\varepsilon > 0$, $\exists \delta > 0$ s.t

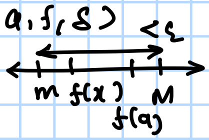
$$M(a,f,\delta) - m(a,f,\delta) < \varepsilon$$

for x s.t $|x-a| < \delta$:

$m(a,f,\delta) \leq f(x), f(a) \leq M(a,f,\delta)$

$\Rightarrow |f(x) - f(a)| < \varepsilon$

$\Rightarrow$ f is cont at a



Theorm: let $A \subseteq \mathbb{R}^n$, is closed. Then $B := \{x \in A \mid o(f,x) \geq \varepsilon\}$ is closed, fix $\varepsilon > 0$.
(f is a bounded function)

proof:

let $U = \mathbb{R}^n \setminus B$. we want to show that $U \subseteq \mathbb{R}^n$ is open. i.e, given $x_0 \in U$, $\delta > 0$ s.t

$$B_\delta(x_0) \subseteq U$$

if $x_0 \in U$ then:

① $x_0 \notin A \Rightarrow x_0 \in \mathbb{R}^n \setminus A$, so $\exists \delta > 0$ s.t

$$B_\delta(x_0) \subseteq \mathbb{R}^n \setminus A$$
$$\Rightarrow B_\delta(x_0) \subseteq U$$

② $x_0 \in A \Rightarrow o(f,x_0) < \varepsilon$ (as $x_0 \notin B$)

$\Rightarrow \exists \varepsilon_1$ s.t

$$o(f,x_0) < \varepsilon_1 < \varepsilon$$

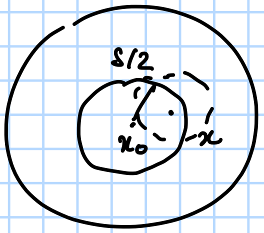
$\Rightarrow \exists \delta > 0$ s.t

$$M(x_0,f,\delta) - m(x_0,f,\delta) \leq \varepsilon_1$$

now if $x \in B_{\frac{\delta}{2}}(x_0)$ then

$$B_{\frac{\delta}{2}}(x) \subseteq B_\delta(x_0)$$

as



$\Rightarrow M(x, f, \delta/2) \leq M(x_0, f, \delta)$

$\& \ m(x, f, \delta/2) \geq m(x_0, f, \delta)$

$\Rightarrow M(x, f, \delta/2) - m(x, f, \delta/2) \leq \varepsilon_1$

$\Rightarrow o(f, x) \leq \varepsilon_1 < \varepsilon$

or

$\forall \ x \in B_{\frac{\delta}{2}}(x_0)$, then

$o(f, x) < \varepsilon$

$\Rightarrow B_{\delta/2}(x_0) \subseteq O$

Differentiation:

We say that $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at point $a \in \mathbb{R}^n$ if $\exists$ a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ s.t

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - T(h)|}{|h|} = 0$$

Then the linear trasformation is called

$Df(a) := T$

is called derivative of f at a

Theorm: If $f: \mathbb{R}^n \to \mathbb{R}^m$ and $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear trasformation s.t

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - T(h)|}{|h|} = 0 \text{ then}$$

T is unique.

proof: suppose there are two linear trasformations, $T_1, T_2: \mathbb{R}^n \to \mathbb{R}^m$ satisfying

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - T_i(h)|}{|h|} = 0$$

← triagle inequality

$$\text{then} \quad \frac{|T_1(h) - T_2(h)|}{|h|} \leq \frac{|f(a+h) - f(a) - T_1(h)|}{|h|} + \frac{|f(a+h) - f(a) - T_2(h)|}{|h|}$$

$\to 0$ as $h \to 0$

$$\Rightarrow \lim_{h \to 0} \left| \frac{T_1(h) - T_2(h)}{h} \right| = 0$$

for $x \neq 0$, $x \in \mathbb{R}^n$, $tx \to 0$ as $t \to 0$

$$\Rightarrow \lim_{t \to 0} \frac{|T_1(tx) - T_2(tx)|}{|tx|} = \frac{1}{|x|} \lim_{t \to 0} \frac{\cancel{|t|}\,|T_1(x) - T_2(x)|}{\cancel{|t|}}$$

$= 0$

$\Rightarrow |T_1(x) - T_2(x)| = 0$

$\Rightarrow T_1(x) = T_2(x) \quad \forall \ x \in \mathbb{R}^n$

$\Rightarrow T_1 \equiv T_2$

Example: $f: \mathbb{R}^2 \to \mathbb{R}$

$(x, y) \longmapsto \sin x$

claim: f is diffiable and $Df(a,b)(x,y) = (\cos a)x$

↳ diff of f at $(a,b) \in \mathbb{R}^2$

we have to show: $\lim\limits_{(h,k)\to(0,0)} \dfrac{|f(a+h, b+k) - f(a,b) - (\cos a)(h)|}{|(h,k)|}$

from 1- calculus

$$\left.\begin{aligned} &= \lim_{(h,k)\to(0,0)} \frac{|\sin(a+h) - \sin(a) - \cos(a)h|}{|h|} \\ &= 0 \end{aligned}\right\} \textcircled{*}$$

$\dfrac{|\sin a \overset{\nearrow 0}{\cos h} + \cos a \overset{\nearrow 0}{\sin h} - \sin a - \cos a h|}{|h|}$

$\Rightarrow$ as $|h| \leq \sqrt{h^2+k^2} = |(h,k)|$

$$\Rightarrow \lim_{(h,k)\to(0,0)} \frac{|\sin(a+h) - \sin(a) - \cos a . h|}{|(h,k)|} \leq \textcircled{*} \to 0$$

$$\Rightarrow \lim_{(h,k)\to(0,0)} \frac{|f(a+h, b+k) - f(a,b) - T(h,k)|}{|(h,k)|} = 0$$

$\Rightarrow Df(a,b)(x,y) = \cos(a).x$

Note: The matrix of linear trasformation $Df(a)$ wrt standard basis from $\mathbb{R}^n \to \mathbb{R}^m$ is denoted by $f'(a)$ and is called the Jacobian matrix of f at a. → $m \times n$ matrix

Ex: $f: \mathbb{R}^n \to \mathbb{R}^m$ is diff at $a \in \mathbb{R}^n$, then f is cont at a.

proof: we want to show $\lim\limits_{h\to 0} |f(a+h) - f(a)| = 0$

let $Df(a) = T$

$$|f(a+h) - f(a)| \leq \underbrace{\frac{|f(a+h) - f(a) - T(h)|}{|h|}}_{\to 0} . |h| + \underbrace{|T(h)|}_{\leq M|h| \to 0 \text{ as } h \to 0}$$

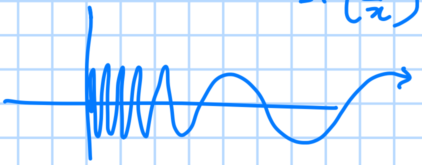
$\Rightarrow \lim\limits_{h\to 0} |f(a+h) - f(a)| = 0$ i.e f is cont at a.

Theorm: (chain rule)

If $f: \mathbb{R}^n \to \mathbb{R}^m$ is diff at a and $g: \mathbb{R}^m \to \mathbb{R}^p$ is differentiable at $b = f(a)$, then $g \circ f: \mathbb{R}^n \to \mathbb{R}^p$ is diff at a, and $D(g \circ f)(a) = Dg(f(a)) \circ Df(a)$

14th Jan:

Exa: $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x)=\begin{cases} 0 & ; x\le 0 \\ \sin(\frac{1}{x}) & ; x>0\end{cases}$



$$o(f,0)=\lim_{\delta\to 0} M(0,f,\delta)-m(0,f,\delta)$$
$$=1-1=2$$

say $f:\mathbb{R}^n\to\mathbb{R}^m$ is defined at $a\in\mathbb{R}^n$ if $\exists$ $T:\mathbb{R}^n\to\mathbb{R}^m$

s.t $\displaystyle\lim_{h\to 0}\frac{|f(a+h)-f(a)-T(h)|}{|h|}=0$

Note: $\exists$ T s.t
$h\longmapsto f(a+h)-f(a)$ or this is very well approximated "upto first order in h". $\left(\frac{T(h)}{|h|}\right.$ exist as $\left.h\to 0\right)$

Eg: Taylor series is similar:
$f(x+a)=f(a)+f'(a).x$ + remainder

Eg: $f:\mathbb{R}\to\mathbb{R}$
$x\longmapsto x^2$

$Df(a)=?$ $\quad Df(a):\mathbb{R}\to\mathbb{R}$

$h\longmapsto f(a+h)-f(a)$

$(a+h)^2-(a)^2$

$\approx 2ah$

$h\longmapsto 2ah$

$Df(a)(h)=2ah$

Jacobian of $f'(a)=[2a]$

Theorm: (chain rule)
If $f:\mathbb{R}^n\to\mathbb{R}^m$ is diff at a and $g:\mathbb{R}^m\to\mathbb{R}^p$ is differentiable at $b=f(a)$, then $g\circ f:\mathbb{R}^n\to\mathbb{R}^p$ is diff at a, and $D(g\circ f)(a)=Dg(f(a))\circ Df(a)$

proof:

$$\lim_{h\to 0}\frac{|f(a+h)-f(a)-Df(a)(h)|}{|h|}=0 \qquad \left(D(g\circ f)(a)=Dg(f(a))\circ Df(a)\right)$$

i.e $\displaystyle\lim_{h\to 0}\frac{|Q(h)|}{|h|}=0$

and $\displaystyle\lim_{k\to 0}\frac{|g(f(a)+k)-g(f(a))-Dg(f(a))(k)|}{|k|}=0$

$\displaystyle\Rightarrow \lim_{k\to 0}\frac{|\psi(k)|}{|k|}=0$

we want to show:

$\displaystyle\lim_{h\to 0}\frac{|\xi(h)|}{|h|}=0$

$$\xi(u) = g\circ f(a+u) - g\circ f(a) - \underbrace{Dg(f(a))}_{T_2}\circ \underbrace{Df(a)}_{T_1}(u)$$

$$T_1(u) = f(a+u) - f(a) - \varphi(u)$$

now

$$|\xi(u)| = |g\circ f(a+u) - g\circ f(a) - T_2\circ\big(f(a+u) - f(a) - \varphi(u)\big)|$$

$$\leq |g\circ f(a+u) - g\circ f(a) - T_2\circ(f(a+u) - f(a))| + |T_2\circ\varphi(u)|$$

$$\Rightarrow \frac{|\xi(u)|}{|u|} \leq \underbrace{\frac{|g\circ f(a+u) - g\circ f(a) - T_2\circ(f(a+u) - f(a))|}{|u|}}_{①} + \underbrace{\frac{|T_2\circ\varphi(u)|}{|u|}}_{②}$$

$$①: \frac{|g(f(a+u)) - g(f(a)) - T_2(f(a+u) - f(a))|}{|f(a+u) - f(a)|} \to 0 \quad . \underbrace{\frac{|f(a+u) - f(a)|}{|u|}}$$

$$\frac{|f(a+u) - f(a)|}{|u|} \leq \overset{\nearrow 0 \text{ as } u \to 0}{\frac{|f(a+u) - f(a) - T_1(u)|}{|u|}} + \frac{|T_1(u)|}{|u|} \leq \frac{M\cancel{|u|}}{\cancel{|u|}}$$

$$\Rightarrow ①: \text{is s.t } ① \leq 0 \times \big(\underset{\substack{\uparrow \\ M \text{ or bounded}}}{M}\big) \Rightarrow ① = 0$$

$$②: \frac{|T_2(\varphi(u))|}{|u|} \leq M' \frac{|\varphi(u)|}{|u|} \to 0 \text{ as } \frac{|\varphi(u)|}{|u|} \to 0$$

so, $D(g\circ f) = D(g(f))\circ D(f)$

Theorm: (I) If $f: \mathbb{R}^n \to \mathbb{R}^m$ is a constant function i.e $\exists$ some $b \in \mathbb{R}^m$ s.t $f(x) = b \ \forall x \in \mathbb{R}^n$ then $Df(a) = 0 \ \forall a \in \mathbb{R}^n$

(II) If $f: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then

$$Df(a) = f \text{ at all } a \in \mathbb{R}^n$$

(III) If $f: \mathbb{R}^n \to \mathbb{R}^m$ then, $f = (f^1, \dots f^m)$ where $f^i: \mathbb{R}^n \to \mathbb{R}$

then f is diff at $a \in \mathbb{R}^n$ iff each $f^i: \mathbb{R}^n \to \mathbb{R}$ is diff at a

also, $\underbrace{Df(a)}_{\mathbb{R}^n \to \mathbb{R}^m} = (\underset{\mathbb{R}^n \to \mathbb{R}}{Df^1(a)}, \dots Df^m(a))$, in terms of matrices

$$f'(a) = \begin{bmatrix} (f^1)'(a) \\ \vdots \\ (f^m)'(a) \end{bmatrix}_{m\times n}$$

proof: (1) $\lim_{h \to 0} \frac{|f(a+h) - f(a) - 0(h)|}{|h|} = 0$

$$= \lim_{h \to 0} \left| \frac{b - b - 0}{h} \right| = 0$$

so f is diff at a and $Df(a) = 0$ $\forall a \in \mathbb{R}^n$

(2) f is given to be a linear transformation

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - f(h)|}{|h|} = \frac{|f(a+h) - f(a+h)|}{|h|} = 0$$

(3) ($\Rightarrow$) Here if f is diff at a then

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$\exists\ T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ for a s.t

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - T(h)|}{|h|} = 0$$

$f^i = \pi^i \circ f \leftarrow$ given diff

$\nwarrow$ ith projection function

$(x^1, \dots x^m) \to x^i$

and it is linear

$\therefore$ diff

by chain rule:

f^i is diff at a.

16th Jan:

Theorem: (iii) If $f : \mathbb{R}^n \to \mathbb{R}^m$ then,
$f = (f^1, \dots f^m)$ where
$f^i : \mathbb{R}^n \to \mathbb{R}$
then f is diff at $a \in \mathbb{R}^n$ iff each $f^i : \mathbb{R}^n \to \mathbb{R}$ is diff at a

also, $\underbrace{Df(a) = (\underbrace{Df^1(a)}_{\mathbb{R}^n \to \mathbb{R}}, \dots Df^m(a))}_{\mathbb{R}^n \to \mathbb{R}^m}$, in terms of matrices

$$f'(a) = \begin{bmatrix} (f^1)'(a) \\ \vdots \\ (f^m)'(a) \end{bmatrix}_{m \times n}$$

proof: (⇒) Here if f is diff at a then

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$\exists\ T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ for a
s.t

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - T(h)|}{|h|} = 0$$

$f^i = \pi^i \circ f \leftarrow$ given diff
$\nwarrow$ ith projection function
$(x^1, \dots x^m) \to x^i$
and it is linear
$\therefore$ diff

by chain rule:

f^i is diff at a.

as $D(f^i) = D(\pi^i \circ f)$

$$= \underset{\substack{\uparrow \\ \text{diff as linear map}}}{D(\pi^i(f))} \circ \underset{\substack{\uparrow \\ \text{diff}}}{D(f)}$$

(⇐) $$\lim_{h \to 0} \frac{|f(a+h) - f(a) - (Df^1(a), \dots Df^n(a))|}{|h|}$$

$$= \lim_{h \to 0} \frac{\left| \sum_{i=1}^{n} (f^i(a+h) - f^i(a) - Df^i(a) \right|}{|h|}$$

this is $|x| \leq \sum |x^i|$

$$\leq \lim_{h \to 0} \sum_{i=1}^{n} \frac{| f^i(a+h) - f^i(a) - D f^i(a) |}{|h|}$$

$= 0$ as each f^i is differentiable

Theorm:(4) If $s:\mathbb{R}^2 \to \mathbb{R}$ is defined by $s(x,y) = x+y$ then $Ds(a,b)=s$

proof : Notice that s is linear, so we can just use Theorm (2)

Theorm :(5) If $p:\mathbb{R}^2 \to \mathbb{R}$ is defined by $p(x,y) = xy$ then

$$Dp(a,b)(x,y) = bx + ay$$

proof :
$$\lim_{(h,k)\to(0,0)} \frac{\left| p(a+h, b+k) - p(a,b) - \overset{= T(h,k)}{(bh+ak)} \right|}{|(h,k)|}$$

$$= \lim_{(h,k)\to(0,0)} \frac{\left|(a+h)(b+k) - (ab) - bh - ak\right|}{|(h,k)|}$$

$$= \lim_{(h,k)\to(0,0)} \frac{|hk|}{|(h,k)|} \leqslant \frac{h^2+k^2}{\sqrt{h^2+k^2}} = \sqrt{h^2+k^2} \to 0$$

$|hk| \leqslant |h|^2 + |k|^2$
as $|h| \leqslant |k|$ or
$|k| \leqslant |h|$

Theorm : If $f, g : \mathbb{R}^n \to \mathbb{R}$ are diff then

① $D(f+g)(a) = Df(a) + Dg(a)$

② $D(f\,g)(a) = g(a)Df(a) + f(a)Dg(a)$

③ If $g(a) \neq 0$, then

$$D\left(\frac{f}{g}\right) = \frac{g(a)Df(a) - f(a)Dg(a)}{(g(a))^2}$$

proof : ① $f+g : \mathbb{R}^n \to \mathbb{R}$

$$f + g = s \circ (f, g)$$

$(f,g) : \mathbb{R}^n \to \mathbb{R}^2$
$x \mapsto (f(x), g(x))$

$s : \mathbb{R}^2 \to \mathbb{R}$
$(z,w) \mapsto z + w$

$$f + g = \underbrace{\underset{\substack{\uparrow \\ \text{diff by theorm (4)}}}{s} \circ (f, g)}_{\text{diff by chain rule}} \leftarrow \text{diff by theorm (3)}$$

$$D(f+g) = D(S \circ (f,g))$$

$$= \overbrace{D(S(f,g))}^{s} \circ D(f,g)$$

$$= S(f,g) \circ D(f,g)$$

$$= S[D(f,g)]$$

$$= S[Df, Dg]$$

$$= Df(a) + Dg(a)$$

② $f \circ g = p \circ (f,g)$ ← done

partial derivatives :

If $f : \mathbb{R}^n \to \mathbb{R}$
$a = (a^1, \dots a^n) \in \mathbb{R}^n$, then

$$\lim_{h \to 0} \frac{f(a^1, \dots a^i + h, \dots a^n) - f(a^1, \dots a^n)}{h} \quad h \in \mathbb{R}$$

(if it exist) is called the i^{th} partial derivative of f at a, and is denoted by $D_i f(a)$

Note: If $g(x) = f(a^1, \dots x, \dots a^n)$ then $(g : \mathbb{R} \to \mathbb{R})$ and then $D_i f(a) = g'(a^i)$

Theorm: let $A \subset \mathbb{R}^n$, If maximum (or minimum) of $f : A \to \mathbb{R}$ occurs at a point a in the interval of A and $D_i f(a)$ exist then

$$D_i f(a) = 0$$

proof :

$g_i(x) = f(a^1 \dots x, \dots a^n)$
then
$g_i(x)$ has a max/min at $x = a^i$
and $g_i \in I \to \mathbb{R}$
↖ some open interval containg a^i $\subseteq \mathbb{R}$

$\Rightarrow 0 = g'_i(a^i) = D_i f(a)$

Exe: If $f, g : \mathbb{R}^n \to \mathbb{R}$ are diff then

① $D(fg)(a) = g(a)Df(a) + f(a)Dg(a)$

① If $g(a) \neq 0$, then

$$D\left(\frac{f}{g}\right) = \frac{g(a)Df(a) - f(a)Dg(a)}{(g(a))^2}$$

as $P(x,y) = xy$

then

$DP(a,b)(x,y) = bx + ay$

$$\text{now } D(f\cdot g) = D(\underbrace{P\circ(f,g)}_{\text{this is diff}})$$

$$= D(P\circ(f,g))\circ D(f,g)$$

$$= D(P\circ(f,g))\circ(Df, Dg)$$

$$= g\cdot Df + f\cdot Dg$$

$$D(f/g) = D(f \times 1/g) = D\left(\frac{1}{g}\right) f + D(f)\,\frac{1}{g}$$

$$= D(g^{-1})\, f + D(f)\,\frac{1}{g}$$

$$= -1 \times D(g)\,\frac{1}{g^2} f + D(f)\,\frac{1}{g}$$

$$= \frac{D(f)\,g - D(g)\,f}{g^2}$$

20th Jan :

Recap : ① Partial derivatives

$$D_i f = \lim_{h \to 0} \frac{f(a^1, \dots a^i + h, \dots a^n) - f(a^1, \dots a^n)}{h}$$

$f: \mathbb{R}^n \to \mathbb{R}$ ⇝ This means that $D_i f(a)$ only if $f: \mathbb{R}^n \to \mathbb{R}$

$D_i f(a)$ ↪ partial derivatie at i^{th} position

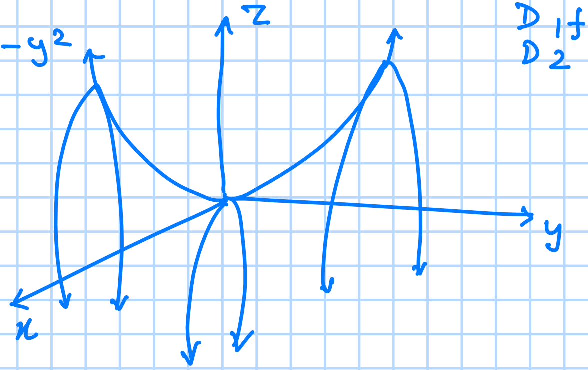
if $g_i(x) = f(a^1, \dots x, \dots a^n)$

$g_i'(a^i) = D_i f(a)$

② Theorm : $A \subset \mathbb{R}^n$, If the maximum or minimum of $f: A \to \mathbb{R}$ occ at a in the interior of A, and if $D_i f(a)$ exist then

$D_i f(a) = 0$

Eg : $f: \mathbb{R}^2 \to \mathbb{R}$
$(x, y) \mapsto x^2 - y^2$

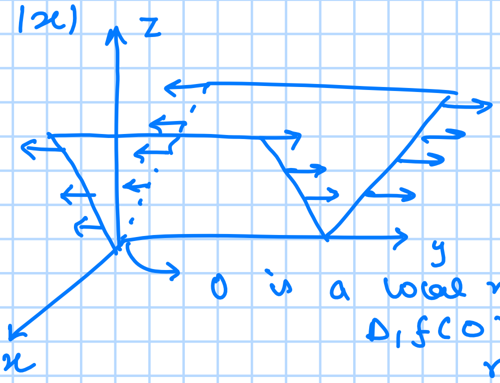


$D_1 f(0) = 0$
$D_2 f(0) = 0$

but 0 is not a point of maximum or minimum

$f_i(0) = 0 \not\Rightarrow f$ is max/min at 0

Eg : $f: \mathbb{R}^2 \to \mathbb{R}$
$(x, y) \mapsto |x|$



0 is a local min but $D_1 f(0)$ does not exist

partial derivative :

If $D_i f(x)$ exist for all $x \in \mathbb{R}^n$, then this defines a function $D_i f: \mathbb{R}^n \to \mathbb{R}$

The j^{th} partial derivative of the function $D_i f$, i.e $D_j(D_i f)$ denoted by
$D_{i,j} f$

Defn : $D_{i,j} f$ is called a second order (mixed) partial derivative of f.

Theorm : If $D_{i,j} f$ and $D_{j,i} f$ are cont in an open set containig a, then

$D_{i,j} f(a) = D_{j,i} f(a)$

proof : later in integration

Similarly we can define third order partial derivative or more generally, partial derivative of order r, $r = 1, 2 \ldots$

Derivative :

Theorem: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, then $D_j f^i$ exist for $1 \leq i \leq m$, $1 \leq j \leq n$ where $(f^1, f^2, \ldots \ldots f^m) = f$

$$f'(a) = m \times n \text{ matrix } \left[D_j f^i(a) \right]_{m \times n}$$

here $f^i : \mathbb{R}^n \to \mathbb{R}$ ∴ legitimate

Proof : First suppose $m = 1$ so $f : \mathbb{R}^n \to \mathbb{R}$, let

j^{th} derivative $f : \mathbb{R}^n \to \mathbb{R}$

$h : \mathbb{R} \to \mathbb{R}^n$

$h(x) = (a^1, \ldots \underset{j^{th} \text{ position}}{x}, \ldots a^n)$ ← linear

then, $D_j f(a) = (f \circ h)'(a^j)$

$h : \mathbb{R} \to \mathbb{R}^n$

$f : \mathbb{R}^n \to \mathbb{R}$ $(m = 1)$

$$\begin{aligned} D_j f(a) &= f'(h(a^j)) \cdot h'(a^j) \quad \text{(chain rule)} \\ &= \underset{1 \times n}{f'(a)} \cdot \underset{n \times 1}{h'(a^j)} \\ &= \underbrace{f'(a)}_{\substack{\mathbb{R}^n \to \mathbb{R}^m \\ \text{then } (f^i)'(a)}} \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th} \text{ position} \end{aligned}$$

$$h'(a^j) = \frac{d}{dx} \begin{pmatrix} a_1 \\ \vdots \\ x \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$(f \circ h)'(a^j)$ has the single entry $D_j f(a)$.

This shows us that $D_j f(a)$ exist and is the j^{th} entry of the $1 \times n$ matrix $f'(a)$.

For arbitrary m, by theorem (3) each f^i is diff (because f is diff) and i^{th} row of $f'(a)$ is $(f^i)'(a)$

so

$$\underset{\substack{\uparrow \\ j^{th} \text{ partial} \\ \text{derivative}}}{D_j} f(a) = \underbrace{\begin{bmatrix} (f^1)'(a) \\ \vdots \\ (f^m)'(a) \end{bmatrix}}_{m \times n} \cdot \underset{n \times 1}{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix}} \leftarrow j^{th} = \underset{m \times 1}{\begin{bmatrix} D_j(f^1)(a) \\ D_j(f^2)(a) \\ \vdots \\ D_j(f^m)(a) \end{bmatrix}}$$

$$D f(a) = \begin{bmatrix} D_1 f(a) & D_2 f(a) & D_3 f(a) & \cdots & D_n f(a) \end{bmatrix}$$

$m \times n$

$$D f(a) = \begin{bmatrix} D_1 f^1(a) & \cdot & \cdot & - & - \\ D_1 f^2(a) & \cdot & - & - & \\ \vdots & & & & \\ D_1 f^m(a) & \cdot & \cdot & \cdot & D_n f^m(a) \end{bmatrix}_{m \times n}$$

$$\left([D_j f^i(a)]_{m \times n} \right)$$

i: 1 to m

j: 1 to n

Note : In general, converse is false, but it is true if we impose additional assumptions (see HW)

Theorm : If $f:\mathbb{R}^n \longrightarrow \mathbb{R}^m$, if all $D_j f^i(x)$ exist in an open set containing a, and if $D_j f^i$ is cont at a then
(D_j f^i is also cont)
$Df(a)$ exist and

$$Df(a) = \left[D_{\underset{1,2\ldots n}{j}} f^{\overset{1,2\ldots m}{i}}(a)\right]_{m\times n}$$

proof : If $f:\mathbb{R}^n \longrightarrow \mathbb{R}$

$$f(a+h) - f(a) \quad \left(f(a+h) - f(a) = \sum_{j=1}^{n} h_j D_j f(c_j)\right)$$

$$= f(a^1+h^1, a^2, \ldots a^n) - f(a^1, a^2, \ldots a^n)$$
$$+ f(a^1+h^1, a^2+h^2, \ldots a^n) - f(a^1+h^1, a^2, \ldots a^n)$$
$$+ \quad \vdots$$
$$+ f(a^1+h^1, \ldots a^n+h^n) - f(a^1+h^1, \ldots a^{n-1}+h^{n-1}, a^n)$$

as $D_j f$ exist $\Rightarrow$ f is cont as a function of x^j and so, mean value theorm tells us that

here

$\sum h_j D_j f(a)$ should be $1\times n$ L.T as $f:\mathbb{R}^n \longrightarrow \mathbb{R}$

$$f(a^1+h^1, \ldots a^n) - f(a^1, a^2, a^3, \ldots, \ldots a^n)$$
$$= h^1 D_1 f(b^1, a^2, \ldots a^n) = h^1 D_1 f(c_1)$$

for some $b^1 \in (a^1, a^1+h^1)$

$f:\mathbb{R}^n \longrightarrow \mathbb{R}$
$D_j f:\mathbb{R} \longrightarrow \mathbb{R}$

then
$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - \sum h^j D_j f(a)|}{|h|}$$
$$= \lim_{h\to 0} \frac{|\sum_j h^j D_j f(c_j) - \sum h^j D_j f(a)|}{|h|}$$
$$\leq \lim_{h\to 0} \sum \frac{|h_j|}{|h|} \underbrace{|D_j f(c_j) - D_j f(a)|}_{\to 0}$$
as $h \to 0$

as $D_j f$ is cont at a.

$$\sum h^j D_j f(a) = \underset{1\times n}{\begin{pmatrix} D_1 f \ldots \end{pmatrix}} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_j \end{pmatrix} = T(h)$$

now technically
$$\underset{1\times n}{\left[D_j f^{\overset{=1}{i}}(a)\right]}$$
$$\downarrow$$
$$[D_j f(a)] = T$$
so $T(h) = \sum h_j D_j f(a)$

if $f:\mathbb{R}^n \to \mathbb{R}^m$ then do same for f^i $\forall$ $i = 1 \ldots m$

Defn : For $A \overset{\text{open}}{\subseteq} \mathbb{R}^n$ and $f: A \longrightarrow \mathbb{R}^m$, then if $D_j f^i(a)$ exist $\forall j, i$ and $\forall$ $x \in A$ is cont on A, then we say f is cont differentiable, or we say

f is of class C^1 on A.

Note : If the partial derivative of the function f^i of order $\leq r$ exist and continuous on A, we say f is of class C^r on A.

Note : If all the partial derivative of the function f^i of all order are continious on A, we say that f is of class C^∞ on A.

Theorm : Let $g_1, \dots g_m : \mathbb{R}^n \to \mathbb{R}$ be continiously differentiable at a. Let $f : \mathbb{R}^m \to \mathbb{R}$ be differentiable at $(g_1(a), \dots g_m(a))$. Define $F : \mathbb{R}^n \to \mathbb{R}$ by $F(x) = f\circ(g_1(x), \dots\dots g_m(x))$, then

$$D_j F(a) = \sum_{j=1}^{m} D_j f(g_1(a), \dots g_m(a)) D_i g_j(a)$$

21st Jan :

Recap : f is diff ⇒ partial derivative of f exist

f is diff ⇐ partial derivative of f exit in an open sets ∋ a and cont on a.

f is of class C^r

Theorm : Let $g_1, \dots g_m : \mathbb{R}^n \to \mathbb{R}$ be continiously differentiable at a. Let $f : \mathbb{R}^m \to \mathbb{R}$ be differentiable at $(g_1(a), \dots g_m(a))$. Define $F : \mathbb{R}^n \to \mathbb{R}$ by $F(x) = f \circ (g_1(x), \dots\dots g_m(x))$, then

$$D_i F(a) = \sum_{j=1}^{m} D_j f(g_1(a), \dots g_m(a)) D_i g_j(a)$$

proof : $F = f \circ g$

$= f \circ (g_1, g_2 \dots g_m$

g_i is cont diff ⇒ g_i are differentiable and hence

g is differentiable

$$F = f \circ g$$

$$\underbrace{F'(a)}_{1 \times n} = \underbrace{f'(g(a))}_{1 \times m} \circ \underbrace{g'(a)}_{m \times n}$$

(as $f : \mathbb{R}^n \to \mathbb{R}^m$

$Df = [D_i f^j]_{m \times n}$ j changes row wise

i changes column wise

matrix multiplication

$$\underset{(D_1 F(a) \cdots D_n F(a))}{F'(a)} = (D_1 f(g(a)) \dots D_m f(g(a))) \begin{pmatrix} D_1 g^1(a) & \cdots \\ D_1 g^2(a) & \cdots \\ \vdots & \\ D_1 g^m(a) & \cdots D_n g^m(a) \end{pmatrix}$$

$$D_j F(a) = \sum_{i=1}^{m} D_j g^i(a) \, D_i f(g(a))$$

j from 1 to n (j^{th} entry)

23rd Jan :

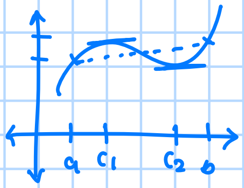
Recap : For funtions $f : \mathbb{R} \to \mathbb{R}$

mean value theorm : suppose f is cont on $[a,b]$ and diff on (a,b) then $\exists$ $c \in (a,b)$ s.t

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or

$$f(b) - f(a) = f'(c)(b-a)$$



Ref : Fleming / Funtion of several variable

Taylor's theorm : Suppose $f : [a,b] \to \mathbb{R}$, suppose $f^{(n-1)}$ is cont on $[a,b]$ and $f^{(n)}$ exist for all points in (a,b) s.t

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \cdots$$

generisation $\rightarrow$ of f for f^n

$$\cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + \frac{f^{(n)}(c)}{n!}(b-a)^n$$

Now, we want to generise this for funtion in many variable.

proposition : Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by $\phi(t) = f(a+th)$. If f is differentiable at $a+th$, then $(f : \mathbb{R}^n \to \mathbb{R})$

$$\phi'(t) = Df(a+th).h$$

proof : Chain rule

$$\phi = f \circ \theta$$

$$\theta : \mathbb{R} \to \mathbb{R}^n$$
$$t \mapsto a + th = \begin{pmatrix} a^1 + th^1 \\ \vdots \\ a^n + th^n \end{pmatrix}$$

θ is differentiable as partial derivative of θ exist and is cont everywhere

$\left(\theta \text{ is diff as } \exists\, T : \mathbb{R} \to \mathbb{R}^n \text{ s.t } T(t) = \begin{pmatrix} h^1 \\ h_2 \\ \vdots \\ h^n \end{pmatrix} \text{ whe } \lim\limits_{\substack{\gamma \to 0 \\ \gamma \in \mathbb{R}}} \frac{|\theta(a+\gamma) - \theta(a) - T(\gamma)|}{|\gamma|} = 0 \right)$

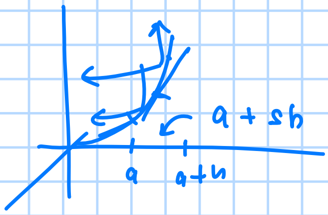
$$D(\phi) = D(f \circ \theta)$$
$$= D(f(\theta))\, D(\theta)$$
$$= D(f(a+th))\; D(\theta)$$
$$= D(f(a+th)) \cdot \begin{bmatrix} D\theta^1 \\ D\theta^2 \\ \vdots \\ D\theta^n \end{bmatrix}$$
$$= D(f(a+th)) \cdot \underset{\overset{\parallel}{h}}{\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}} = D(f(a+th)).h$$

Theorm : (Mean value theorm in several variables) Let $f : \mathbb{R}^n \to \mathbb{R}$ be diff at every point of the line segment joining a and $a+h$

then $\exists$ $s \in (0,1)$ s.t

$$f(a+h) - f(a) = Df(a+sh).h$$

proof : let $\phi:\mathbb{R}\to\mathbb{R}$ given by



$$\phi(t)=f(a+th)$$

by MVT (1-variable), $\exists\ s\in(0,1)$ s.t

$$\phi(1)-\phi(0)=\phi'(s)$$

$$\Rightarrow f(a+h)-f(a)=D(f(a+sh))\cdot h$$

(By our proposition)

(This is the directional derivative of f at a point a+sh, in the direction of h)

($x,y\in K \Rightarrow tx+(1-t)y\in K\ \forall\ t\in[0,1]$)

Corr: let $K\subseteq\mathbb{R}^n$ be open convex set and let $f:K\to\mathbb{R}$ be differentiable. let $c\geq 0$ be a number s.t $|f'(x)|\leq c\ \forall x\in K$. Then for every $x,y\in K$, we have

$$|f(x)-f(y)|\leq c|x-y|$$

proof: We mean value theorm with $a=y$ and $a+h=x$

$$f(x)-f(y)=Df(y+s(x-y))\cdot(x-y) \quad \text{for some } s\in(0,1)$$

$$\Rightarrow |f(x)-f(y)| = |\underbrace{Df(y+s(x-y))}\cdot \underset{\in\mathbb{R}^n}{(x-y)}|$$

as $Df(y+s(x-y))$ is $1\times n$ matrix $\quad 1\times n$ matrix $\quad (\ )\in\mathbb{R}^n$

$\to [f'(y+s(x-y))]^T$ is $n\times 1$ matrix $\in\mathbb{R}^n$

dot product

$$\Rightarrow |f(x)-f(y)| = |\langle (f'(y+s(x-y)))^T,(x-y)\rangle|$$

$$\leq |f'(y+s(x-y))||(x-y)| \quad \text{(cauchy swarz)}$$

$$\leq c|x-y|$$

Theorm: (Taylor's theorm in several variables) let $f:A\to\mathbb{R}$ where A is open and $A\subseteq\mathbb{R}^n$.

let $f:A\to\mathbb{R}$ be of class C^q. let $a,x\in A$ s.t line segment joining a and x is contained in A.

Then:

$$f(x)=f(a)+\sum_{i=1}^{n}D_i f(a)(x^i-a^i)$$

$$+\frac{1}{2!}\sum_{i,j=1}^{n}D_{i,j}f(a)(x^i-a^i)(x^j-a^j)+\dots\frac{1}{(q-1)!}\sum_{i_1,i_2,\dots i_{q-1}=1}^{n}D_{i_1,i_2,\dots i_{q-1}}f(a)\overset{h_{i_1}}{(x^{i_1}-a^{i_1})}\cdots(x^{i_{q-1}}-a^{i_{q-1}})$$

$$+R_q(x)$$

$\exists\ s \in (0,1)\quad R_q(x) = \frac{1}{q!} \sum_{i_1, i_2 \ldots i_q = 1}^{n} D_{i_1, i_2 \ldots i_q} f(a+sh) \cdot (h^{i_1})(h^{i_2}) \ldots (h^{i_q})$

proof:

let $h = x - a$, $\phi(t) = f(a+th)$

using our proposition we get:

$$\phi'(t) = D(f(a+th)) \cdot h$$

$$= \sum_{i=1}^{n} D_i (f(a+th))(h^i)$$

another application of proposition:

$$\phi''(t) = \sum_{i,j=1}^{n} D_{i,j} (f(a+th))(h^i)(h^j)$$

$$\vdots$$

$$\phi^q(t) = \sum_{i_1, i_2, \ldots i_q = 1}^{n} D_{i_1, i_2 \ldots i_q} (f(a+th))(h^{i_1})(h^{i_2}) \ldots (h^{i_q})$$

taylor's theorm in 1-variable:

$$\phi(1) = \phi(0) + \phi'(0) + \frac{\phi''(0)}{2!} + \cdots + \frac{\phi^{q-1}(0)}{(q-1)!} + \frac{\phi^q(s)}{q!}$$

for some $s \in (0,1)$

$\phi(1) = f(x)$

$\phi(0) = f(a)$, by substitution we are done.

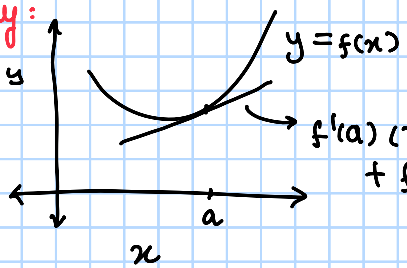
27^{th} Jan:

Recap: mean value theorm, taylor's theorm

Note: $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \underbrace{f(a) + f'(a)(x-a)}_{\text{polynomial function of } x} + R_2(x)$$

geometrically:



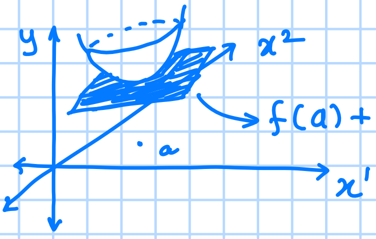
$\to f'(a)(x-a)$ graph $+ f(a)$

Taylors theorm for function of several variable:

$$f(x) = \underbrace{f(a) + \sum_{i=1}^{n} D_i f(a)(x^i - a^i)}_{\substack{\text{Polynomial of degree 1} \\ \text{in the variables } x^1, x^2 \dots x^n}} + R_2(x)$$

Eg: $n = 2$

$f : \mathbb{R}^2 \to \mathbb{R}$



$\to f(a) + D_1 f(a)(x^1 - a^1) + D_2 f(a)(x^2 - a^2)$

(Tangent plane)

Inverse functions:

In tut-2, $f : \mathbb{R}^n \to \mathbb{R}^n$ is diff and has diff inverse

then

$$\underset{\mathbb{R}^n \to \mathbb{R}^n}{f(f^{-1}(x)) = x}$$

$f' \to$ Jacobian

$Df \to \mathbb{R}^n \to \mathbb{R}^m$

$$\underset{\substack{n \times n \\ \text{jacobian matrix}}}{(f^{-1}(a))'} = \Big[\underset{\substack{n \times n \\ \text{j. matrix}}}{(f(f^{-1}(a)))'} \Big]^{-1}$$

($f' \to n \times m$ jacobian matrix
$Df \to$ transformation)

Note: In particular, f is diff with diff inverse at all $b \in \mathbb{R}^n$

$\Rightarrow f'(b)$ is invertible $\forall\, b \in \mathbb{R}^n$ (This is from linear algebra)

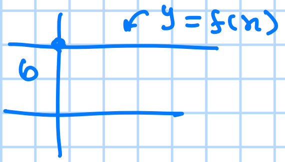
Eg: $f : \mathbb{R} \to \mathbb{R}$
$x \mapsto x$
$f(x) = x \qquad f^{-1}(x) = x$

$f : \mathbb{R}^n \to \mathbb{R}^n$
$x \mapsto x$
$f(x) = x$
$f^{-1}(x) = x$

Eg: $f: \mathbb{R} \longrightarrow \mathbb{R}$

$f(x) = ax + b$ for some fixed $a, b \in \mathbb{R}$

what if $a = 0$



not 1-1
not onto } so f is not invertible

for $a \neq 0$
and any b

$f(x) = ax + b$
is invertible

Eg: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$f(x) = Ax$

when A is invertible
i.e $\det(A) \neq 0$

$f(x) = Ax$ is an invertible function

$f^{-1}(x) = A^{-1}x$

$f'(a) = A$

$Df(a)(x) = Ax$

$Df^{-1}(a)(x) = A^{-1}(x)$

Eg:

$f: \mathbb{R} \rightarrow \mathbb{R}$

$f(x) = x^3$

$f^{-1}(x) = x^{1/3}$

at $x = 0$

$f'(x)$ exist even tho f is not diff invertible at 0

but

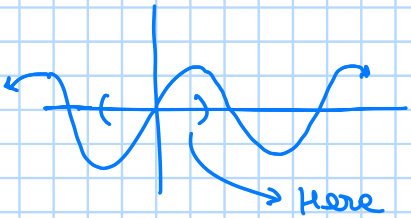
$(f^{-1})'(x)$ does not

Exe: given $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ how can we determine if f is invertible?

Exe: Suppose we know f is invertible, how to determine weather f^{-1} is differentiable?

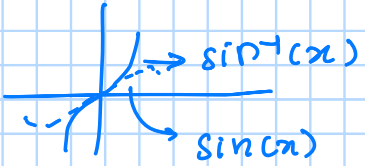
These questions will be answered using the inverse function theorm.

Exe: $f(x) = \sin(x)$
this is not invertible for $\mathbb{R} \rightarrow \mathbb{R}$



Here the function is invertible

special domain "close to 0"



Theorm: (Inverse fuction theorm for $f: \mathbb{R} \to \mathbb{R}$) suppose $f: \mathbb{R} \to \mathbb{R}$ of class C^1 and $f'(a) \neq 0$ then f is locally invertible around a, and local inverse is f^{-1} is diff.

Proof: ① $f'(a) \neq 0 \Rightarrow f'$ is non-zero near a

$\Rightarrow f$ is locally monotone (so 1-1)

$\Rightarrow f$ is locally invertible

① $f \in C^1(\mathbb{R})$
② $f'(a) \neq 0$

$\Rightarrow f$ is locally invt arond a & f^{-1} is diff

② Image under f of a small neighbourhood of a is open (IVT)

③ $U \to$ open $\Rightarrow f(U)$ is open then

$\overset{X}{U \subset X} \Rightarrow f(U) \overset{Y}{\subset Y}$ is open

$f(U)$ is open given U is open

$\Rightarrow f^{-1}(x)$ is cont

④ by MVT, $\exists c \in (b, f^{-1}(f(b)+h))$

$$f'(c) = \frac{f(f^{-1}(f(b)+h)) - f(b)}{f^{-1}(f(b+h))-b}$$

$$= \frac{f(b)+h-f(b)}{f^{-1}(f(b)+h)-b}$$

$$= \frac{h}{f^{-1}(f(b)+h)-b}$$

(MVT for $(b, f^{-1}(f(b)+h)$ then $\frac{1}{f'(c)} \to \frac{1}{f'(b)}$ as f' is C^1 and $f'(b) \neq 0$)

here f is C^1 or f' is cont

$$\Rightarrow \lim_{h\to 0} \frac{f^{-1}(f(b)+h)-b}{h} = \frac{1}{f'(b)} \quad \text{as } h \to 0,\ c \to b$$

$$\Rightarrow \lim_{h\to 0} \frac{f^{-1}(y+h) - f^{-1}(y)}{h} = \frac{1}{f'(f^{-1}(y))}$$

Note: Note that the conclusion is false if f in not C^1

$$f(x) = \begin{cases} x + 2x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\Big(f(x) = x + 2x^2 \sin\left(\frac{1}{x}\right)$

$f'(x) = 1 + 4x \sin\left(\frac{1}{x}\right) - 2\cos\left(\frac{1}{x}\right) \Big)$ at $x \to 0$ this is not defd

f is not C^1

this is not locally invertible near 0

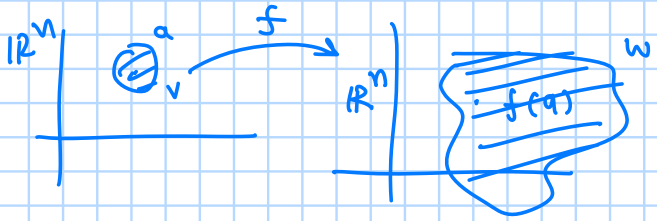
$\cos\left(\frac{1}{x}\right) \to \frac{1}{2}$ for $\frac{1}{x} = 2\pi n + \frac{\pi}{6}$

$x = \frac{1}{2\pi n + \frac{\pi}{6}}$ $n \to \infty$

for $x \to 0$ s.t $\cos(1/x) \to 1/2$ then $f'(x_n) = 0 \to f(x_n)$ is cont

Theorm: (Inverse function theorm) Suppose $f: \mathbb{R}^n \to \mathbb{R}^n$ is C^1 is an open set containing a, and let $\det(f'(a)) \neq 0$, then $\exists$ open set V contang a and an open set W cont $f(a)$ s.t

$f: V \to W$ has a continious inverse $f^{-1}: W \to V$ which is diff $\forall\, y \in W$, $(f^{-1})'(y) = [f'(f^{-1}(y))]^{-1}$



$\mathbb{R}^n$
a
V
f
$\mathbb{R}^n$
f(a)
W

28th Jan.

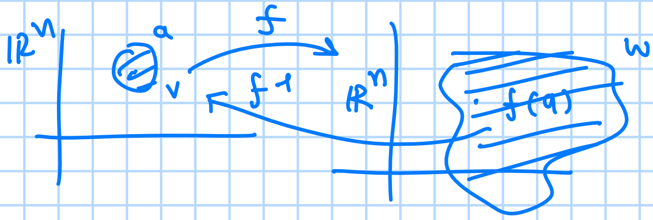
Theorm : (Inverse function theorm) Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 is an open set containing a, and let $\det(f'(a)) \neq 0$, then $\exists$ open set V containg a and an open set W cont $f(a)$ s.t

invertible

$f : V \to W$ has a continious inverse $f^{-1} : W \to V$ which is diff $\forall\, y \in W$, $(f^{-1})'(y) = [f'(f^{-1}(y))]^{-1}$

why do we care?

① Solving (system) of equations



Eg : Claim for all c suff close to 1, $\exists x$ s.t

$$x + \frac{e^{\sin(x)}}{100} - \frac{e^{\sin(1)}}{100} = c$$

observe $x = 1$ satisfies

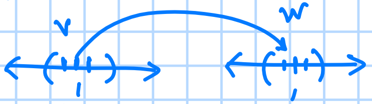
$f(x) = 1$

$$1 + \frac{e^{\sin(1)}}{100} - \frac{e^{\sin(1)}}{100} = 1$$

$$f'(x) = 1 + \frac{e^{\sin x}}{100} \cos(x)$$

$$f'(1) = 1 + \frac{e^{\sin(1)}}{100} \cos(1) > 0$$

$\neq 0$



① f is C^1
② $f'(1) \neq 0$ $\Big\} \Rightarrow$ f is locally invertible

so, $\exists x$ close to 1 s.t

$$x + \frac{e^{\sin(x)}}{100} - \frac{e^{\sin(1)}}{100} = f(x) = c$$

for $c \in W$, $f^{-1}(c)$ exist
$\Rightarrow$ x solution as $f(x) = c$

Lemma : let $A \subseteq \mathbb{R}^n$ be an open rectangle and let $f : A \to \mathbb{R}^n$ be continiously differentiable, then if $\exists M$ (a number)

s.t $|D_j f^i(x)| \leq M \;\forall x, y \in A$

then

$|f(x) - f(y)| \leq n^2 M |x - y|$

proof :

$$|(f^i)'(x)| = |(D_1 f^i(x), D_2 f^i(x), \ldots D_n f^i(x))|_{1 \times n}$$

$$\leq \sum_{j=1}^{n} |D_j f^i(x)| \leq nM$$

$$\Rightarrow |f(x) - f(y)| \leq \sum_{i=1}^{n} |f^i(x) - f^i(y)| \leq \sum_{i=1}^{n} nM|x - y| = n^2 M |x - y|$$

from lemma after mean value theorm

(that lemma $|f'(x)| \leq M \Rightarrow |f(x) - f(y)| \leq M(x - y)$ but $f : \mathbb{R}^n \to \mathbb{R}$)

proof: (Inverse function theorm)

let T be the linear trasformation Df(a), then det(f'(a)) ≠ 0 inplies that T is non-singular (invertible)

$f:\mathbb{R}^n \to \mathbb{R}^n$ (Derivative of linear transformation)

$$\underset{n\times n}{D(T^{-1}\circ f)} = \underbrace{DT^{-1}(f(a))}_{n\times n} \circ \underbrace{Df(a)}_{T} = T^{-1}\circ T = I_{n\times n}$$

($\exists T$ so $\exists T^{-1}$ as $\det(T) \neq 0$)

If the theorm is true for $g = T^{-1}\circ f$ then it is true for f

wlog we asume $Df(a) = I_d$

(Normal usage of base as T is invertible or $T\circ g = f$
$\uparrow$ invertible $\uparrow$ invertible)

so, if $f(a+h) = f(a)$ then

$$\frac{|f(a+h) - f(a) - T(h)|}{|h|} = \frac{|h|}{|h|} = 1$$

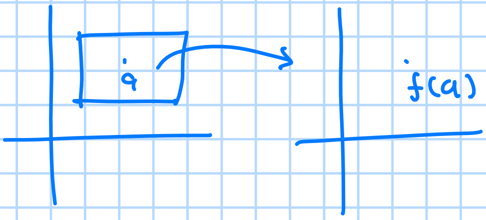
but as f is diff: (we can sense one-one here)

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - h|}{|h|} = 0$$

⇒ ∃ an closed rectagle U s.t

$a \in \text{Interior}(U)$
$f(x) \neq f(a)$
$\forall x \in U,\ x \neq a$

(as if $f(a+h) = f(a)$ then contradiction)



since f is continiously diff in an open set containing a, by making U smaller, we can assume that

$\det(f'(x)) \neq 0$ for all $x \in U$

$\Rightarrow |D_j f^i(x) - D_j f^i(a)| < \frac{1}{2n^2}$ for some $n \in \mathbb{N}$

comes from the continuity of derivative

measure of closeness

$\Rightarrow |D_j f^i(x) - \delta_{i,j}| < \frac{1}{2n^2}$

$i = j \Rightarrow \delta_{i,j} = 1$ else 0

now ① f is one-to-one on U

$|D_j g^i(x)| < \frac{1}{2n^2}$

$g(x) = f(x) - x$

$\Rightarrow |D_j g^i(x)| = |D_j f^i(x) - \delta_{ij}| < \frac{1}{2n^2}$

known

$$\Rightarrow \underbrace{|(f(x_1) - x_1) - (f(x_2) - x_2)|}_{|g(x_1) - g(x_2)|} \leq n^2 \cdot \frac{1}{2n^2} |x_1 - x_2|$$

By lemma

$$\Rightarrow |g(x_1) - g(x_2)| \leq \frac{1}{2} |x_1 - x_2|$$

$$|f(x_1) - x_1 - f(x_2) + x_2| \leq \frac{1}{2} |x_1 - x_2|$$

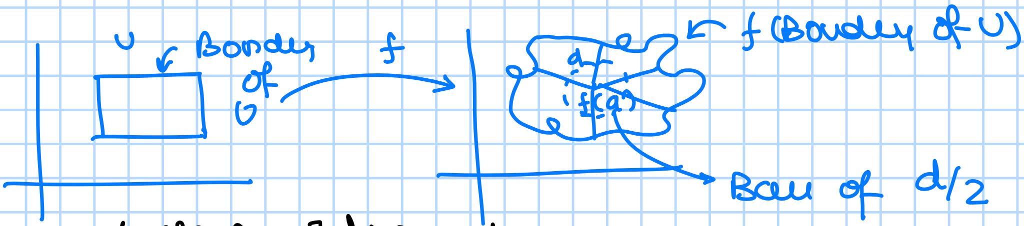
$$\Rightarrow -|f(x_1) - f(x_2)| + |x_1 - x_2| \leq \frac{1}{2} |x_1 - x_2|$$

$$\Rightarrow |x_1 - x_2| \leq 2 |f(x_1) - f(x_2)|$$

so f is 1-1 on U $\left(f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \right)$

② image under f of a neighborhood of a in U, contains an open ball around $f(a)$

f(boundary U) is a compact set as boundary of U is a compact set as f is cont



therefore $\exists d > 0$ s.t

$$|f(x) - f(a)| \geq d \quad \forall x \in \text{Boundry of } U$$

(from week 1)

let $W = \left\{ y \,\middle|\, |y - f(a)| < \frac{d}{2} \right\}$

if $y \in W$ and $x \in$ Boundry of U then

$$\left.\begin{array}{l} |y - f(a)| < d/2 \\ |y - f(x)| \geq d/2 \end{array}\right\} \Rightarrow \underset{(*)}{|y - f(a)| < |y - f(x)|}$$

③ we will show that for any $y \in W$, $\exists$ unique $x \in \text{int}(U)$ s.t $f(x) = y$

consider $g : U \to \mathbb{R}$ by

$$g(x) = |y - f(x)|^2 = \sum_{i=1}^{n} |y^i - f^i(x)|^2$$

$g(x)$ is cont as f is cont and y is fixed

as U is closed then is minim of g in U.

as $g(a) < g(x)$ from $(*)$ for $x \in$ Bondry U

g does not attain its maxima on bondry.

$$\Rightarrow \exists x_0 \in \text{interior}(U) \text{ s.t } g(x_0) \geq g(x) \quad \forall x \in U$$

$$\Rightarrow D_j g(x_0) = 0 \quad \forall j$$

$$\Rightarrow \sum 2 (y^i - f^i(x_0)) \underbrace{D_j f^i(x_0)}_{} = 0$$

↙ Non Zero

and $D_jf^i(x_0)$ has non-zero determinant (By chosen U)

$$\Rightarrow \begin{pmatrix} y^1 - f^1(x_0) \\ \vdots \\ y^n - f^n(x_0) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Rightarrow y = f(x_0) \Rightarrow W \subseteq f(U)$$

if $V = \text{int}(U) \cap f^{-1}(W)$

then

V is open as $f^{-1}(W)$ is open

and $f : V \longrightarrow W$

has an inverse

$f^{-1} : W \rightarrow V$

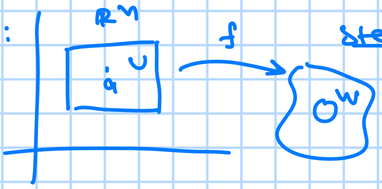
① one - one

③ onto

30th Jan :

Theorm : (Inverse function theorm) Suppose $f:\mathbb{R}^n \to \mathbb{R}^n$ is C^1 in an open set containing a, and let $\det(f'(a)) \neq 0$, then ∃ open set V containg a and an open set W cont f(a) s.t $f: V \to W$ has a continious inverse (invertible) $f^{-1} \cdot W \to V$ which is diff $\forall\, y \in W$, $(f^{-1})'(y) = [f'(f^{-1}(y))]^{-1}$

last time we proved :



step 1 : f is one-one on U using $x_1, x_2 \in U$ $\Rightarrow |x_1 - x_2| \leqslant 2|f(x_1) - f(x_2)|$ ① ———

step-2 : To show that the image under f of a neighbourhood of a in U contains an open ball around f(a).

$$\underbrace{\text{interior}(U) \cap f^{-1}(W)}_{\text{open}} = \overset{\uparrow}{V} \quad \text{is open}$$

$f: V \to W$ is one-one and onto

proof : Step 3 : f^{-1} is cont :

$|f^{-1}(y_1) - f^{-1}(y_2)| \leqslant 2|y_1 - y_2| \quad \forall\, y_1, y_2 \in W$ (this is already proved)

($y_1 = f(x_1)$, $y_2 = f(x_2)$, for some $x_1, x_2 \in V$ then the inequality ① gives us this)

$\Rightarrow f^{-1}$ is continious.

Step 4 : f^{-1} is differentiable :

let $y \in W$, $y = f(x)$ for some $x \in V$ (what is a guess for $Df^{-1}(y)$)

our guess: $(Df(x))^{-1}$ $\quad Df^{-1}(y) = Df^{-1}(f(x))$

(as $y = f(x)$ guess: $Df^{-1}(y) = (Df(x))^{-1}$)

let $T_1 = Df(x)$

as f is diff :

$f(x_1) = f(x) + T_1(x_1 - x) + Q(x_1 - x)$ (this is by def of diff)

s.t $\lim\limits_{x_1 \to x} \dfrac{|Q(x_1 - x)|}{|x_1 - x|} = 0$ ——— ②

$\Rightarrow T_1^{-1}(f(x_1) - f(x)) = (x_1 - x) + T_1^{-1}(Q(x_1 - x))$

(By applying T^{-1} on both sides)

$T_1^{-1}(y_1 - y) = (f^{-1}(y_1) - f^{-1}(y)) + T_1^{-1}(Q(f^{-1}(y_1) - f^{-1}(y)))$

($f^{-1}(y) = x$)

now, $[f^{-1}(y_1) - f^{-1}(y)] - [T_1^{-1}(y_1 - y)] = -T_1^{-1}(Q(f^{-1}(y_1) - f^{-1}(y)))$

we have to show:

$$\lim_{y_1 \to y} \frac{|T_1^{-1}(Q(f^{-1}(y_1)-f^{-1}(y)))|}{|y_1-y|} = 0$$

now $\exists\, M_1 > 0$ s.t $|T_1^{-1} z| \leqslant M_1 |z|$
$\forall z \in \mathbb{R}^n$

$$\Rightarrow \lim_{y_1 \to y} \frac{|T_1^{-1}(Q(f^{-1}(y_1)-f^{-1}(y)))|}{|y_1-y|} \leqslant M \frac{|Q(f^{-1}(y_1)-f^{-1}(y)|}{|y_1-y_2|}$$

$$= M \underbrace{\frac{|Q(f^{-1}(y_1)-f^{-1}(y))|}{|f^{-1}(y_1)-f^{-1}(y)|}}_{\substack{\longrightarrow 0 \\ \text{by } ②}} \cdot \underbrace{\frac{|f^{-1}(y_1)-f^{-1}(y)|}{|y_1-y|}}_{\substack{\leqslant 2 \\ \text{by } ①}}$$

$$\Rightarrow \lim_{y_1 \to y} \frac{|T_1^{-1}(Q(f^{-1}(y_1)-f^{-1}(y)))|}{|y_1-y|} \overset{\leqslant 0}{=} 0$$

so $Df^{-1}f(x) = [Df(x)]^{-1}$

3rd Feb :

please attempt Spivak Q 2.37(a)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be cont diff. show that f is not 1-1

(Hint: If $D_1 f(x,y) \neq 0 \ \forall \ (x,y) \in A$ look at $g : A \to \mathbb{R}^2 \quad g(x,y) = (f(x,y), y)$ draw a graph)

$$g : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x,y) \mapsto (f(x,y), y)$$

Implicitly defined functions:

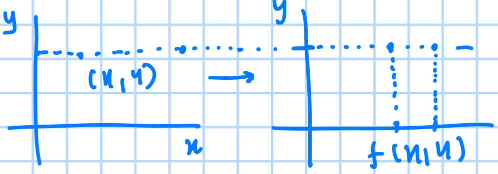
Eg: $x^3 y + 2e^{xy} = 0$ defines y as a diff function of x: to find $\frac{\partial y}{\partial x}$

$$\begin{bmatrix} D_1 f(x,y) & \cdot \\ 0 & 1 \end{bmatrix}$$

$$3x^2 y + x^3 \frac{\partial y}{\partial x} + 2e^{xy}\left(y + x\frac{dy}{dx}\right) = 0$$

$$\det(g'(x,y))$$
$$= \det(D_1 f(x,y))$$

$$\frac{dy}{dx} = \frac{-(2ye^{xy} + 3x^2 y)}{(x^3 + 2xe^{xy})}$$



if we can write $y = g(x)$ then

$$x^3 + 2xe^{xy} \neq 0$$

if $g(x_1,y_1) = g(x_2,y_2)$
$\Rightarrow x_1 = x_2 , y_1 = y_2$
so if $f(x_1,y_1) = f(x_2,y_2)$
& $y_1 = y_2 \Rightarrow x_1 = x_2$
but in a small region $D_1 f \neq 0 \Rightarrow x_1 \neq x_2$ *

More generally if $f(x,y) = 0$ and this determines y as a function of x say $y = g(x)$ then:

$$\left(f(\phi(x)) = 0 \quad \phi(x) \mapsto (x, g(x)) \right)$$

$$f(x, g(x)) = 0$$

by chain rule:

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} 1 \\ g'(x) \end{bmatrix} = 0$$

$$\left(f(x, g(x)) = 0 \quad \begin{bmatrix} D_1 f & D_2 f \end{bmatrix} \cdot D(x, g(x)) \quad \mathbb{R} \to \mathbb{R}^2 \right)$$

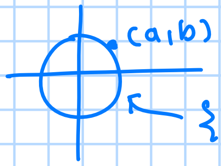
$$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot g'(x) = 0$$

$$\Rightarrow g'(x) = \frac{-\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$$

if g exist then $\frac{\delta f}{\delta y} \neq 0$

Note: we will see later that $\frac{\partial f}{\partial y} \neq 0$ is a sufficient condition

Eg: $f(x,y) = x^2 + y^2 - 1$



$\{ (x,y) \mid f(x,y) = 0 \}$

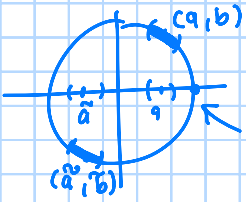
$$f(x,y) = 0 \Leftrightarrow x^2 + y^2 = 1$$
$$\Rightarrow y = \pm \sqrt{1 - x^2}$$

for (a,b) w/ $f(a,b)=0$

$b>0$ $\frac{\partial f}{\partial y}\neq 0$

$y=\sqrt{1-x^2}$

for x in neighbourhood of a



for $(\tilde{a},\tilde{b})$ $\tilde{b}<0$, $\frac{\partial f}{\partial y}\neq 0$

$y=-\sqrt{1-x^2}$

issue here at $(1,0)$

$\frac{\partial f}{\partial y}=0$ $y=g(x)$ not written for a neighborhood of x

Note: let $x\in\mathbb{R}^n$, $y\in\mathbb{R}^m$, let C be an $m\times(m+n)$ matrix

$$C=[L_{m\times n}\quad M_{m\times m}]_{m\times(m+n)}$$

let us denote

$$f:\mathbb{R}^n\times\mathbb{R}^m\longrightarrow\mathbb{R}^m$$

$$f(x,y)=C\begin{bmatrix}x^1\\ \vdots\\ x^n\\ y^1\\ \vdots\\ y^m\end{bmatrix}_{(m+n)\times 1}$$

$$=L_{m\times n}\,x_{n\times 1}+M_{m\times m}\,y_{m\times 1}$$

for x,y s.t $f(x,y)=0$ when can we have $y=g(x)$

$$f(x,y)=0$$
$$\Rightarrow Lx+My=0$$
$$\Rightarrow My=-Lx$$
$$\Rightarrow y=-M^{-1}Lx$$

for M is invertible we have

$$y=-M^{-1}Lx$$
$$=g(x)$$

Ex: given $f:\mathbb{R}^n\times\mathbb{R}^m\longrightarrow\mathbb{R}^m$ and $a\in\mathbb{R}^n$, $b\in\mathbb{R}^m$ s.t $f(a,b)=0$, where can we write y as a function of x, for y near b? i.e $y=g(x)$ $(g:\mathbb{R}^n\rightarrow\mathbb{R}^m)$ s.t $f(x,g(x))=0$ for x near a?

Note: $f:\mathbb{R}^k\longrightarrow\mathbb{R}^p$ when $k>p$ then here $k-p$ vaibles wrt p Variables we want to find so we have to solve same for $\mathbb{R}^{k-p}\times\mathbb{R}^p\longrightarrow\mathbb{R}^p$
(x,y)

Note: If we have $f(x,y)=c$ then $f_1(x,y)=x^2+y^2$

we want

$$f_1(x,y)=1$$
$$\Rightarrow f(x,y)=\underbrace{x^2+y^2-1}_{\text{even tho } f_1(x,y)=c}=0$$

Theorm: (Implicit function theorm) suppose $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is continiously diffentiable in a open set containing (a,b) and $f(a,b)=0$. let M be the $m \times m$ matrix:

$$\left[D_{n+j} f^i(a,b) \right]_{n \times n} \quad 1 \leq i, j \leq m$$

$$\left(f'(a,b) = [\, L \quad M \,] \right)$$

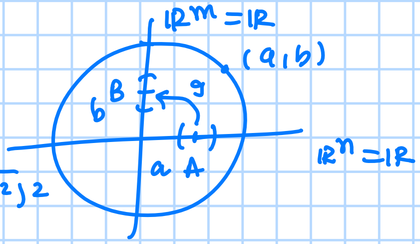
If $\det M \neq 0$, there is an open set $A \subseteq \mathbb{R}^n$ containg a and open set $B \subseteq \mathbb{R}^m$ containig b, with the following property for each $x \in A$ there is a unique $g(x) \in B$ s.t

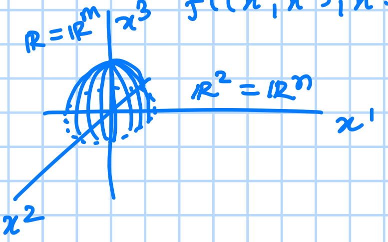
$$f(x, g(x)) = 0$$

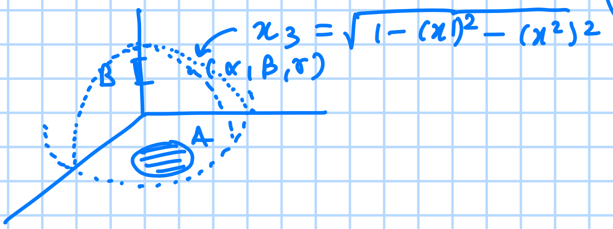
The fuction g is diff

Eg: $f : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ given by

$f((x^1, x^2), x^3) = (x^1)^2 + (x^2)^2 + (x^3)^2 - 1$







proof: Define $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ by $F(x,y) = (x, f(x,y))$

then

$$F(a,b) = (a, f(a,b)) = (a, 0)$$

and

$$F'(x,y) = \begin{bmatrix} \partial x / \partial x & \partial x / \partial y \\ \partial f(x,y) / \partial x & \partial f(x,y) / \partial y \end{bmatrix}$$

$$= \begin{bmatrix} I_{n \times n} & 0_{n \times m} \\ L_{m \times n} & M_{m \times m} \end{bmatrix}$$

$\hookrightarrow m \times (n+m)$ $(m+n) \times (m+n)$

so $\det(F'(a,b)) = \det M \neq 0$

so, Inverse fuction theorm applied to $F \Rightarrow \exists$ open $W \subseteq \mathbb{R}^n \times \mathbb{R}^m$ s.t $F(a,b) = (a,0)$ and $\exists$ open $V \subseteq \mathbb{R}^n \times \mathbb{R}^m$ cont (a,b) s.t

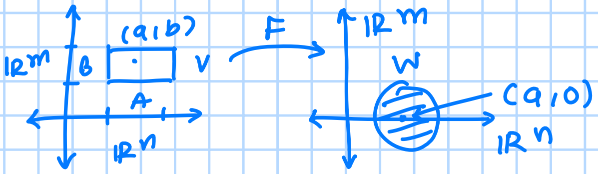
$F : V \longrightarrow W$ has a diff inverse

$h : W \longrightarrow V$

wlog $V = A \times B$

$h(x,y) = (x, k(x,y))$

for some diff fuction k



4th Feb :

Theorm: (Implicit funtion theorm) suppose $f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m$ is continiously differentiable in a open set containing (a,b) and $f(a,b)=0$. let M be the $m\times m$ matrix:

$$[D_{n+j}f^i(a,b)]_{n\times n} \quad 1\le i,j\le m$$

$$\left(f'(a,b)=[L \quad M]\right)$$

If $\det M\neq 0$, there is an open set $A\subseteq\mathbb{R}^n$ containg a and open set $B\subseteq\mathbb{R}^m$ containig b, with the following property for each $x\in A$ there is a unique $g(x)\in B$ s.t

$$f(x,g(x))=0$$

The funtion g is diff

proof: Define $F:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n\times\mathbb{R}^m$ by $F(x,y)=(x,f(x,y))$ then

$$F(a,b)=(a,f(a,b))$$
$$=(a,0)$$

and

$$F'(x,y)=\begin{bmatrix}\partial x/\partial x & \partial x/\partial y\\ \partial f(x,y)/\partial x & \partial f(x,y)/\partial y\end{bmatrix}$$

$$=\begin{bmatrix}I_{n\times n} & O_{n\times m}\\ L_{m\times n} & M_{m\times m}\end{bmatrix}$$

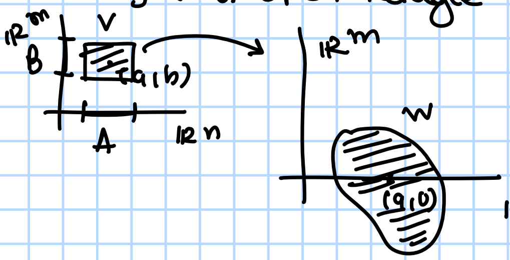
$\hookrightarrow m\times(n+m)$ $(m+n)\times(m+n)$

so $\det(F'(a,b))=\det M\neq 0$

so, Inverse funtion theorm applied to F $\Rightarrow$ $\exists$ open $W\subseteq\mathbb{R}^n\times\mathbb{R}^m$ cont $(a,0)$ s.t $F(a,b)=(a,0)$ and $\exists$ open $V\subseteq\mathbb{R}^n\times\mathbb{R}^m$ cont (a,b)

s.t $F:V\longrightarrow W$ has a diff inverse $h:W\longrightarrow V$ (h is uniquely defined and hence so is K)

wlog V is open retagle



assume $V=A\times B$

$h(x,y)=(x,K(x,y))$ for some diff funtion K

let $\pi:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m$ be defind by

$\pi(x,y)=y$ then

$$\pi\circ F(x,y)=f(x,y)$$
$$\Rightarrow f(x,K(x,y))=f\circ h(x,y)$$

$$\Rightarrow f(x,K(x,y))=(\pi\circ F)\circ h(x,y)$$
$$=\pi\circ(F\circ h)(x,y)$$
$$=\pi\circ(Id)(x,y)$$

$= \pi(x,y)$
$= y$

so $f(x, k(x,0)) = 0$

define $g(x) = k(x,0)$ then

$g : A \longrightarrow B$ satisfies

$f(x, g(x)) = 0$ (k is uniquely defind so is g)

Exe : $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$f(x,y,z) = x^2 + y^2 + z^2 - 1$

$\{(x,y,z) \mid f(x,y,z) = 0\} \longrightarrow$

$f : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$

$(x,y), z \longmapsto f(x,y,z)$

can we do $z = g(x,y)$

$(a,b) = \left(\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}\right)\right)$

$$f'(x,y,z) = [\underbrace{2x \quad 2y}_{L_{1\times2}} \quad \underbrace{2z}_{M_{1\times1}}]$$

$$f'(a,b) = \begin{bmatrix} \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \end{bmatrix}$$

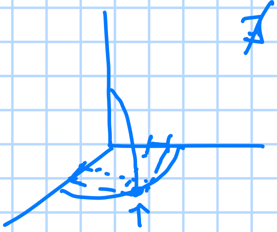
$M = \begin{bmatrix} \frac{2}{\sqrt{3}} \end{bmatrix}$ $\det M = \frac{2}{\sqrt{3}} \neq 0$

so, implicit function tells us $\exists\, g : A^{\subseteq \mathbb{R}^2} \longrightarrow B^{\subseteq \mathbb{R}}$ s.t

$f(x, y, g(x,y)) = 0$

but $M = [0]$ say for $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$\nexists$ any neighbourhood of $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ where we can write z as function of x,y



$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$P = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$f'(P) = \left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$

$= [D_1 f \quad D_2 f \quad D_3 f]$

s.t $D_2 f$ is invertible, we can write y as a function of x and z in a neig of P

Exe: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $f(x,y) = x^2 - y^3$

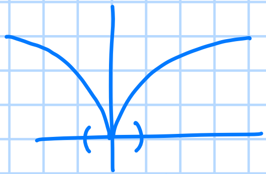
then $(0,0)$ satisfies $f(x,y) = 0$

$$f'(x,y) = [2x \quad -3y^2]$$

$$f'(0,0) = [0 \quad 0]$$

$M = [0]$ does not sat inverse function theorm

but still



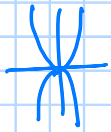
$$y^3 = x^2 \Rightarrow y = (x^2)^{1/3} = x^{2/3}$$

here g is not diff at 0 but still it exist and is unique

Exe: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ $f(x,y) = y^2 - x^4$

$$f'(x,y) = [-4x^3 \quad 2y]$$

$$f'(0,0) = [0 \quad 0]$$



$M = [0]$ det $M = 0$

but still $y = g(x) = \pm x^2$ ← not unique but function exist

6th Feb :

Recap : Implicit function theorm

$$f:\mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$(x, y)$$

(a,b) satisfying $f(a,b)=0$

Note : $y=g(x)$ uniquely for $(x,y)\in\{f=0\}$ near (a,b)

$$f'(a,b)=\Big[\ \ \underbrace{\boxed{M}}_{m\times m}\Big]_{m\times(m+n)}$$

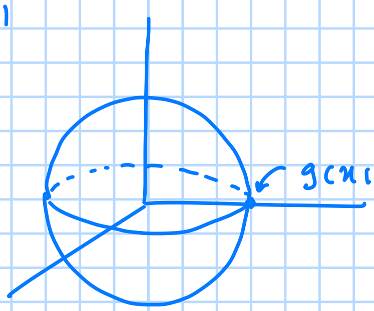
If $\det(M)\neq 0$, then $\exists\, g$ s.t

$$y=g(x)$$

Exe : $f:\mathbb{R}^3 \longrightarrow \mathbb{R}$

$f(x,y,z)=x^2+y^2+z^2-1$

$P=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$



$g(x,y)=\pm\sqrt{1-x^2-y^2}$

$\uparrow$ not unique

Converse of Implicit function theorm

If we can write $y=g(x)$ for points $(x,y)\in\{f=0\}$ is it true that $\det M\neq 0$

Eg : $f:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$

$(x,y)\longrightarrow y^2$

so $f(x,y)=0$ when $y^2=0$

$\Rightarrow y=0$

$\Rightarrow x$-axis

$\Big(f(x,y)=|y-g(x)|^2$ for general cases$\Big)$

so for any point on x-axis we can write $y=g(x)$ uniquely for some function $g(x)=0$

But $f'(x,y)=\begin{bmatrix} D_1f & D_2f\end{bmatrix}_{1\times 2}$

$$=\begin{bmatrix} 0 & 2y\end{bmatrix}_{1\times 2}$$

for $(x,y)\in\{f=0\}$

$$f'(x,y)=\begin{bmatrix} 0 & \underbrace{\boxed{0}}_{M_{1\times 1}}\end{bmatrix}_{1\times 2}$$

$\det(M)=0$

Theorm : let $A \subseteq \mathbb{R}^{n+m}$ be open. Let $f: A \to \mathbb{R}^m$ be diff. write f in the form $f(x,y)$ for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and write

$$f'(x,y) = \left[\underbrace{\frac{\partial f}{\partial x}}_{m \times n} \; \underbrace{\frac{\partial f}{\partial y}}_{m \times m} \right]_{m \times (m+n}$$

$$\frac{\partial f}{\partial x} = \left[D_j f^i \right]_{m \times n} \qquad \frac{\partial f}{\partial y} = \left[D_{n+j} f^i \right]_{m \times m}$$

$1 \leq j \leq n$, $1 \leq i \leq m$ $\qquad$ $1 \leq j \leq m$, $1 \leq i \leq m$

suppose $\exists g : B \to \mathbb{R}^m$ (where $B \subseteq \mathbb{R}^n$ is open) s.t $f(x, g(x)) = 0$ $\forall x \in B$ then for $x \in B$

$$\frac{\partial f}{\partial x}(x, g(x)) + \frac{\partial f}{\partial y}(x, g(x))\, g'(x) = 0$$

so if $\frac{\partial f}{\partial y}(x, g(x))$ is non-singular then

$$g'(x) = -\left[\frac{\partial f}{\partial y}(x, g(x)) \right]^{-1} \frac{\partial f}{\partial x}(x, g(x))$$

Proof : follows from the case of 2D
$f(x, g(x)) = 0$
where $f : \mathbb{R}^{m+n} \to \mathbb{R}^m$
$g : \mathbb{R}^n \to \mathbb{R}^m$

by chain rule:
$\phi(x) = (x, g(x))$
$\phi : \mathbb{R}^m \to \mathbb{R}^{m+n}$
$f(\phi(x)) = 0$

$$\phi'(x) = \begin{bmatrix} I_{m \times m} \\ g'(x)_{n \times m} \end{bmatrix}_{(m+n) \times (m)} \qquad f'(\phi(x)) \circ \phi'(x) = 0$$

$$\Rightarrow \left[\frac{\partial}{\partial x} f(\phi(x)) \; \frac{\partial}{\partial y} f(\phi(x)) \right] \circ \phi'(x) = 0$$

$$\Rightarrow \frac{\partial}{\partial x} f(x, g(x)) + \frac{\partial}{\partial y} f(x, g(x)) \circ g'(x) = 0$$

Theorm : (Rank theorm) let $f : \mathbb{R}^n \to \mathbb{R}^p$ be cont diff in an open set containing a, where $p \leq n$. If $f(a) = 0$ and the $p \times n$ matrix $(D_j f^i(a))$ has rank p, Then there is an open set $A \subseteq \mathbb{R}^n$ cont a and a diff funtion

$h : A \to \mathbb{R}^n$ with differentiable inverse s.t

$$f \circ h(x^1, x^2, \dots x^n) = \underbrace{(x^{n-p+1}, \dots x^n)}_{\text{last } p \text{ coordinates}}$$

$$\left(\begin{array}{l} f : \mathbb{R}^n \to \mathbb{R}^p \\ f'(x) = [\quad]_{p \times n} \text{ if rank } [\quad]_{p \times n} = p \end{array} \right)$$

proof : let $f : \mathbb{R}^{n-p} \times \mathbb{R}^p \to \mathbb{R}^p$

Case I : if $\det(M) \neq 0$ where
$M = (D_j f^i(a)) \leftarrow$ last $p \times p$

then by proof of implicit function theorm

$\exists\, h : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ s.t

$h(x,y) = (x, k(x,y))$

$\pi : \mathbb{R}^{n-p} \times \mathbb{R}^p \longrightarrow \mathbb{R}^p$

$\pi(x,y) = y$

$\pi \circ F(x,y) = \pi \circ (x, f(x,y))$

$= f(x,y)$

$\Rightarrow f(x, k(x,y)) = f \circ h(x,y)$

$\Rightarrow f \circ h(x,y) = (\pi \circ F) \circ h(x,y)$

$= \pi \circ (F \circ h)(x,y)$

$= \pi \circ I(x,y)$

$= \pi(x,y)$

$= y$

(this is from the proof of implicit function theorm)

$\Rightarrow f \circ h(x^1, x^2, \dots x^n) = (x^{n-p+1}, \dots x^n)$

so, $f \circ h(x^1, x^2, \dots x^n) = (x^{n-p+1}, \dots x^n)$

i.e $\underset{\mathbb{R}^p}{f \circ h(x,y)} = \underset{\mathbb{R}^p}{(y)}$

Case II: In general, $(D_j f^i(a))$ has rank p, $\exists\, j_1 \dots j_p$ s.t $p \times p$ matrix

$$\overline{M} = (D_j f^i(a))_{1 \le i \le p,\ j = j_1, \dots j_p}$$

has rank p, i.e $\det \overline{M} \neq 0$

If $g : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ permutes x^j s.t

$g(x^1, \dots x^n) = (\dots, x^{j_1}, \dots x^{j_p})$ $(\tilde{x}, \tilde{y})$

then

$f \circ g(x^1, \dots x^n)$ is a function that we delt with case I

$\exists\, \tilde{h}$ s.t $\tilde{h} : \mathbb{R}^n \longrightarrow \mathbb{R}^p$

s.t $f \circ g \circ \tilde{h}(x,y) = y$

define

$h = g \circ \tilde{h}$

($f \circ g$ = a function s.t

$[\ (\ )\ ]$ $(\ )_{p \times p}$ has det 0, so $\exists\, h$ from case I)

$\rightarrow$ $p \times p$ is invertible

10th Feb:

Integration: (Riemann)

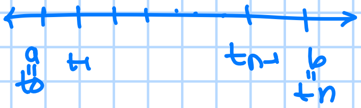
we will define the integral of a function $f : A \to \mathbb{R}$, where A is a rectangle

$$A = [a_1, b_1] \times \dots \times [a_n, b_n]$$

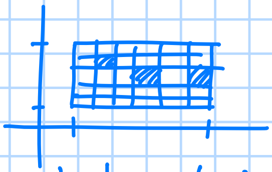
Recall, a partition P of an interval $[a, b]$ is collection of numbers $t_0, t_1, \dots t_k$ s.t

$$a = t_0 \leq t_1 \leq \dots < t_k = b$$

the partition P divides $[a, b]$ into k subintervals

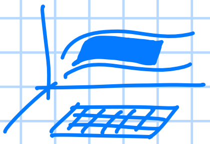


Def: A partition of a rectangle $[a_1, b_1] \times \dots \times [a_n, b_n] \subseteq \mathbb{R}^n$ is a collection P
$P = (P_1, P_2 \dots P_n)$
where each P_i is a partition of the interval $[a_i, b_i]$



Eg: suppose $P_1 = t_0, t_1 \dots t_k$ is a partition of $[a_1, b_1]$ and $P_2 = s_0, s_1 \dots s_l$ is a partition of $[a_2, b_2]$, then $P = (P_1, P_2)$ is a partition of $[a_1, b_1] \times [a_2, b_2]$ which devides it into $(k)(l)$ subrectangles of form $[t_{i-1}, t_i] \times [s_{j-1}, s_j]$

In general, if P_i is a partition of $[a_i, b_i]$ which divides $[a_i, b_i]$ into N_i subintervals then $P = (P_1, \dots P_n)$ is a partition of $[a_1, b_1] \times \dots \times [a_n, b_n]$ which is $\prod N_i$ sub rectangles.



Def: (volume) we define volume of the rectangle $S = [a_1, b_1] \times \dots \times [a_n, b_n]$ to be

upper and lower sum: $v(S) = (b_1 - a_1) \times (\quad) \times \dots \times (\quad) \times (b_n - a_n)$

Suppose A is a rectangle, $f : A \to \mathbb{R}$ a bounded function, and P is a partition of A. For each S (subrectangle) of the partition let

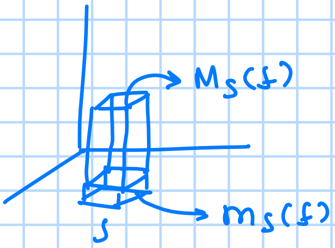
$$m_S(f) = \inf \{ f(x) \mid \forall x \in S \}$$

$$M_S(f) = \sup \{ f(x) \mid \forall x \in S \}$$

Def: (lower and upper sum) upper and lower sum of f for partition P are

$$L(f, P) = \sum_S m_S(f) \, v(S)$$

$$U(f, P) = \sum_S M_S(f) \, v(S)$$



Note: $L(f, P) \leq U(f, P)$

Def: (Refinement) we say the partition P' refines P if every subrectangle of P' is contained in a subrectangle of P.

lemma: Suppose that partition P' refines P, then $L(f,P) \leqslant L(f,P')$ and $U(f,P') \leqslant U(f,P)$

proof: Each subrectangle S of P is divided into subrectangles $S_1, S_2, \dots S_\alpha$ of P'

then, $v(S) = v(S_1) + \dots + v(S_\alpha)$
also $m_S(f) \leqslant m_{S_j}(f)\ \forall j$

$$\Rightarrow v(S_i)\, m_S(f) \leqslant m_{S_i}(f)\, v(S_i)\ \forall i$$

$$\Rightarrow \sum_i v(S_i) m_S(f) \leqslant \sum_i m_{S_i}(f) v(S_i)$$

$$\Rightarrow v(S)\, m_S(f) \leqslant \sum_i m_{S_i}(f) v(S_i)$$

now $\sum_S v(S) m_S(f) \leqslant \sum_S \sum_i m_{S_i}(f) v(S_i)$

$$= \sum_{S'} m_{S'}(f) v(S')$$

summing over all subrectangles

$\Rightarrow L(f,P) \leqslant L(f,P')$

The proof of $U(f,P') \leqslant U(f,P)$ is similar to above.

Cor: If P, P' are any two partitions, then $L(f,P') \leqslant U(f,P)$

proof: Let P'' be a partition which refines both P and P' then
(Eg of $P'' = (P''_1, P''_2, \dots P''_n)$ where P''_j is a refinement of $[a_j, b_j]$ which refines both P_j and P_j')

then $L(f,P') \leqslant L(f,P'')$ $\longrightarrow$ P'' refines P'
$U(f,P'') \leqslant U(f,P)$ $\longrightarrow$ P'' refines P

$\Rightarrow L(f,P') \leqslant L(f,P'') \leqslant U(f,P'') \leqslant U(f,P)$
$\Rightarrow L(f,P') \leqslant U(f,P)$

Note: From above cor we have

$$\sup_P L(f,P) \leqslant \inf_P U(f,P)$$

(supremum over all possible lower sum
infimum over all possible upper sum)

Def: $f: A \rightarrow \mathbb{R}$ is Riemann integrable if

$$\sup_P L(f,P) = \inf_P U(f,P)$$

If this happen then we define this common number to be the integral of f over A, denoted by

$$\int_A f$$

If $f:[a,b] \rightarrow \mathbb{R}$, where $a < b$ then

$$\int_a^b f = \int_{[a,b]} f$$

Theorm: A bounded fuction $f: A \to \mathbb{R}$ is integrable iff $\forall \varepsilon > 0, \exists$ a partition P of A s.t
$$U(f,P) - L(f,P) < \varepsilon$$

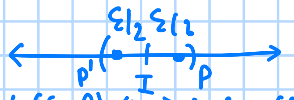
proof:

($\Leftarrow$) Then f is integrable as if not then $\inf_P U(f,P) - \sup_P L(f,P) = a > 0$ some a

this means $\forall P$ $U(f,P) - L(f,P) \geq a$
this is a contradiction

($\Rightarrow$) If f is integrable then, $\exists$ partitions P, P' s.t

$U(f,P) - L(f,P') < \varepsilon$ by defination of supremum and infimum



$\sup_P L(f,P) = \inf_P U(f,P)$

by defination of sup and inf $\exists$ such partitions then

$P'' =$ refinement of P, P''

$\Rightarrow U(f,P) - L(f,P') < \varepsilon$

as $U(f,P'') \leq U(f,P)$
$L(f,P'') \geq L(f,P')$

$\Rightarrow U(f,P'') - L(f,P'') \leq U(f,P) - L(f,P') < \varepsilon$
$\Rightarrow U(f,P'') - L(f,P'') < \varepsilon$

Exe: $f: [0,1] \times [0,1] \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} 0 & ; x \in \mathbb{Q} \\ 1 & ; x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

is f riemman integrable?

$f(x,y) = \begin{cases} 0 & ; x \in \mathbb{Q} \\ 1 & ; x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

now

$f(x,y)$ is bounded (trivial)

also now $m_s(f) = 0$
as for every rectangle say

$[a,b] \times [c,d]$ between a and b
there is a rational:

$(b-a) > 0$
$\hookrightarrow$ some positive quantity

$\lfloor b-a \rfloor = n$
$n \leq b-a < n+1$
$\frac{n}{n+1} \leq \frac{b-a}{n+1} < b-a$

$a < a + \frac{n}{n+1} < a + \frac{b-a}{n+1} < \cancel{a+}b\cancel{-a}$
$\Rightarrow a < a + \frac{n}{n+1} < b$

if a is rational then
$c = a + \frac{n}{n+1}$ is rational

and $a < a + (c-a)\frac{\sqrt{2}}{2} < c$

$\downarrow$

Irrational

if a is irrational then

c = irrational

then

$a < c$

let $z = c - a$

$\exists n$ s.t

$n > \frac{1}{z} \Rightarrow nz > 1$

$\Rightarrow n(c-a) > 1$

$nc - na > 1$

$\exists m$ s.t

$na < m < nc$

$\Rightarrow a < \frac{m}{n} < c$ so $\exists$ a rational

$\therefore$ b/w any two numbers there is a rational and an irrational

$\Rightarrow m_s(f) = 0$

$M_s(f) = 1 \quad \forall s$

now $\sum v(s) m_s(f) = 0$

$\sum v(s) M_s(f) = (1-0)(1-0) = 1$

then $\inf_S M(f,S) \neq \sup_S m(f,S)$

11th feb :

Quiz solution:

Idea $F : \mathbb{R}^2 \to \mathbb{R}^2$

$$F'(x,y) = \begin{bmatrix} 2ye^{2x}(2) & 2e^{2x} \\ e^y & xe^y \end{bmatrix}$$

at point : $\left.\begin{matrix} 2ye^{2x} = 2 \\ xe^y = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} x=0 \\ y=1 \end{matrix}$

$F : V \to W$

$F^{-1} : W \to V$

s.t $f(0,1) = (2,0)$

so $\exists r > 0$ s.t

$B_r(2,0) \subseteq W$

$\Rightarrow f^{-1}(B_r(2,0)) \subseteq V$

Recap : $f : \underset{\nwarrow}{A} \to R$, Partition, upper/lower sums

closed rectangle in $\mathbb{R}^n$

Exe: $f : A \to \mathbb{R}$

be a const function

$f(x) = c$

$\forall P \in \mathcal{P}(A)$, any $S \in P$ ⟶ subrectangle

we have $m_S(f) = M_S(f) = c$

$\Rightarrow L(f,P) = \sum_S m_S(f)\vartheta(S)$

$L(f,P) = c\vartheta(A)$

similarly $U(f,P) = \sum_S M_S(f)\vartheta(S)$

$U(f,P) = c\vartheta(A)$

and so, Supremum

$\sup L(f,P) = c\vartheta(A)$

$\inf U(f,P) = c\vartheta(A)$

$\Rightarrow$ f is riemann integrable

and $\int_A f = c\vartheta(A)$

Exe: let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & ; x \in \mathbb{Q} \\ 1 & ; x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

for any $P \in \mathcal{P}(A)$, for any subrectangle S

$m_S(f) = 0$

as S will contain (x_0, y_0)

with $x_0 \in \mathbb{Q}$

$M_S(f) = 1$

as S will contain (x_1, y_1)

with $x_1 \in \mathbb{R} \setminus \mathbb{Q}$

$\Rightarrow$ $L(f,P) = 0$

$U(f,P) = 1 \times \vartheta([0,1] \times [0,1])$

$= 1$

$\forall P \Rightarrow \inf U(f,P) = 1$ $\Rightarrow$ f is <u>not</u> riemann integrable

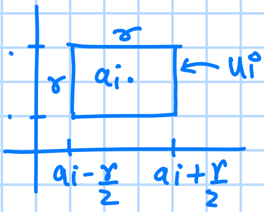
$\sup L(f,P) = 0$

Defn: A subset A of $\mathbb{R}^n$ has (n-dimensional) measure 0 if $\forall \varepsilon > 0$ there is a cover $\{U_1, U_2, \dots\}$ of A by countably many closed rectangles U_i s.t

$$\sum_{i=1}^{\infty} v(U_i) < \varepsilon$$

Exe: A has finitely many points then measure of A = 0

if $A = \{a_1, \dots a_k\}$ with U_i to be closed rectangle contain a_i, $v(U_i) < \frac{\varepsilon}{k}$



$v(U_i) = r^n$
choose r small enough
to get
$r^n < \varepsilon/k$

$$\Rightarrow \sum_{i=1}^{k} v(U_i) < \sum_{i=1}^{k} \frac{\varepsilon}{k} = \varepsilon$$

Note: we allow countably many points, then A has measure 0, because if A is say $A = \{a_1, a_2, \dots\}$, we can choose U_i to be closed rectangle containing a_i s.t $v(U_i) < \varepsilon/2^i$ then

$$\sum_{i=1}^{\infty} v(U_i) < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \frac{\frac{\varepsilon}{2}}{1-\left(\frac{1}{2}\right)} = \frac{\frac{\varepsilon}{\cancel{2}}}{\frac{1}{\cancel{2}}} = \varepsilon$$

this is by power series conv

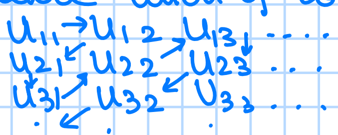
Theorm: If $A = A_1 \cup A_2 \cup A_3 \cdots$ and each A_i has measure 0, then A has measure 0.

proof: say we are given ε,
as each A_i has measure, $A_i = 0$
$\exists$ cover $\{U_{i,1}, U_{i,2}, \dots\}$ of A_i by closed rectangles s.t the sum

$$\sum_{j=1}^{\infty} v(U_{i,j}) < \varepsilon/2^i$$

so $\bigcup_{i \geq 1} \{U_{i,1}, U_{i,2}, \dots\}$ is a countable collection of closed rec

(as countable union of countably many sets



)

write $\bigcup_{i \geq 1} \{U_{i,1}, U_{i,2}, \dots\}$ as $\{V_1, V_2, \dots\}$
then $\sum_{i=1}^{\infty} v(V_j) < \sum_{j=1}^{\infty} \varepsilon/2^j = \varepsilon$

Exe: Prove that $\mathbb{R} \subseteq \mathbb{R}^2$ has measure 0.

$\mathbb{R} \subseteq \mathbb{R}^2$ has measure 0 as
let $\mathbb{R} = \bigcup_{-\infty}^{\infty} \left(\frac{r}{n}, \frac{r+2}{n}\right)$ then

every $\left(\frac{r}{n}, \frac{r+2}{n}\right)$ is inside

$1/n$ then $v(S) = \frac{2}{n^2}$

letting $\frac{2}{n^2} < \frac{\varepsilon}{(2)^r}$ we get

$\sum v(S) < \sum \frac{\varepsilon}{(2)^r} = \varepsilon$, $\therefore$ measure $\mathbb{R}$ in $\mathbb{R}^2 = 0$

Defn: A subset A of $\mathbb{R}^n$ has (n-dimmensional) content 0 if for $\forall \varepsilon > 0$, there is a finite cover of A by closed rectangles

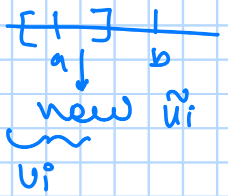
$\{U_1, U_2 \ldots U_K\}$ s.t $\sum_{i=1}^{K} v(U_i) < \varepsilon$ $\left(\underset{<\varepsilon}{\text{finite cover}} \Rightarrow \underset{<\varepsilon}{\text{Countable cover}}\right)$

Note: If A has content 0 $\Rightarrow$ A has measure 0, by defination

Theorm: If $a<b$ then $[a,b] \subset \mathbb{R}$ does not have content 0. If $\{U_1, U_2 \ldots U_n\}$ is a finite cover of $[a,b]$ by closed intervals then

$$\sum_{i=1}^{n} v(U_i) \geq b-a$$

proof: we can assume that each $U_i \subset [a,b]$



let $a = t_0 < t_1 < \cdots < t_K = b$
be the endpoints of the sets U,
then each $v(U_i)$ is the sum of some numbers $(t_j - t_{j-1})$

also each interval $[t_{j-1}, t_j]$ lies in some U_i.

$$\Rightarrow \sum_{i=1}^{n} v(U_i) \geq \sum_{i=1}^{K} (t_j - t_{j-1}) = b-a$$

Theorm: If A is compact and has measure 0 then A also has content 0.

proof:

A has measure 0, $\exists$ cover $\{U_1, \ldots\}$
as A is compact
$\Rightarrow \exists \{U_{K_1}, U_{K_2} \ldots U_{K_n}\}$
s.t this is a finite subcover

as $\sum_{i=1}^{\infty} v(U_i) < \varepsilon$

$$\Rightarrow \sum_{i=1}^{n} v(U_{K_i}) \leq \sum_{i=1}^{\infty} v(U_i) < \varepsilon$$

$$\Rightarrow \sum_{i=1}^{n} v(U_{K_i}) < \varepsilon$$

$\Rightarrow$ A has content 0

Remark: A has content 0 $\Rightarrow$ A has measure 0
converse is not always true
but if A is compact then converse is true.

13th Feb :

Recap : measure 0 (given $\varepsilon > 0$ $A \subseteq \bigcup_{i \geq 1} U_i$ U_i is closed retagles s.t $\sum v(U_i) < \varepsilon$)

content 0 (cover A by finitely many closed retagles)

Note : A compact and A has measure 0 $\Rightarrow$ A has content 0

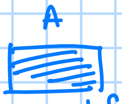
To see : f is (Riemman) integrable iff $\{x \mid f \text{ is not cont at } x\}$ has measure 0

Recall $O(f, x)$ as oscilation of f at x and $\left(O(f,a) = \lim_{\delta \to 0} M(a,f,\delta) - m(a,f,\delta) \right.$ $M(a,f,\delta) = \sup\{f(x) \mid |x-a|<\delta\}$)

(theorm covered) $O(f,x) = 0 \Leftrightarrow f$ is cont at x
and $\forall \varepsilon > 0$, $B_\varepsilon = \{x \mid O(f,x) \geq \varepsilon\}$ is closed (for fixed ε)

Lemma : let A be a closed rectangle and $f : A \to \mathbb{R}$ be a bounded function s.t $O(f,x) < \varepsilon$ for $\forall x \in A$. Then there is a partion P of A, then there is a partion P of A s.t

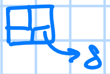
$$U(f,P) - L(f,P) < \varepsilon\, v(A)$$



$f : A \longrightarrow \mathbb{R}$ bounded
$|f(A)| \leq M$

proof : given $x \in A$, $\exists$ a closed rectangle U_x s.t

$x \in \text{int}(U_x)$
and $M_{U_x}(f) - m_{U_x}(f) < \varepsilon$
(as $O(f,x) < \varepsilon$)



$|x - x_0| < \delta$

$O(f,x) < \varepsilon$ $\forall x \in A$
then $\exists P \in \mathcal{P}(A)$ s.t
$U(P,f) - L(P,f) < \varepsilon\, v(A)$

now as A is compact, $\exists$ finite collection of $\{U_{x_1}, U_{x_2}, \dots U_{x_k}\}$ which cover the closed rectangle A (By compatness of A)

let P be partion of A s.t each subretagle S of P is contained in some U_{x_i}

All endpoints make partition

($\forall S \in P$, $\exists U_x$ s.t $S \subseteq U_x$
$M_S(f) - m_S(f) < M_{U_x}(f) - m_{U_x}(f) < \varepsilon$)

now, for any $S \in U_{x_i}$ $M_S(f) - m_S(f) < M_{U_{x_i}}(f) - m_{U_{x_i}}(f) < \varepsilon$

$$\Rightarrow M_S(f) - m_S(f) < \varepsilon$$

$$\Rightarrow U(f,P) - L(f,P) = \sum_S (M_S(f) - m_S(f))\, v(S)$$

$$< \varepsilon \sum_S v(S) = \varepsilon\, v(A)$$

$$\Rightarrow U(f,P) - L(f,P) < \varepsilon\, v(A)$$

Theorm : let A be a closed retagle and let $f : A \to \mathbb{R}$ be a bounded function. $B = \{x \mid f \text{ is not cont at } x\}$ then :

f is integrable
$\Updownarrow$
B is a set of measure 0

proof : ($\Leftarrow$) Suppose that B has measure 0
now we want to show that f is integrable
$B_\varepsilon = \{x \mid O(f,x) \geq \varepsilon\}$
then B_ε is closed & compact (as $O(f,x) = 0$ for cont)

also $B_\varepsilon \subseteq B$

so B_ε has measure 0, is closed and compact

$\Rightarrow$ B_ε has content 0

so $\exists$ finite collection $\{U_1, U_2, \dots U_k\}$ of closed rect which covers B_ε with the sum of

$$\sum_{i=1}^{k} v(U_i) < \varepsilon$$ (this is by defination)

let P be a partition of A s.t

$\forall S \in P$ (some partition)

$\hookrightarrow$ subrectangle

$S \in$ one of two groups

① $S_1 : \{S \mid \text{s.t } S \subseteq U_i \text{ for some } i\}$

② $S_2 : \{S \mid \text{s.t } S \cap U_i = \phi \ \forall i \text{ i.e } S \cap B_\varepsilon = \phi\}$

let $|f(x)| \leq M \quad \forall x \in A$

$\Rightarrow M_S(f) - m_S(f) \leq 2M \quad \forall S$ (By Boundeness)

$\Rightarrow \sum_{S \in S_1} (M_S(f) - m_S(f)) v(S) \leq 2M \sum_{S \in S_1} v(S) \leq 2M \sum_{i=1}^{k} v(U_i)$

$< 2M\varepsilon$

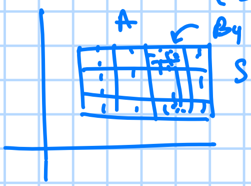
$\left(\sum_{S \in S_1} (M_S(f) - m_S(f)) v(S) < \sum_{U_i} 2M\, v(U_i) < 2M\varepsilon \right)$

and if $S \in S_2$ then $O(f, x) < \varepsilon \quad \forall x \in S$

then by the lemma proved above that implies, there is a refinement P' of P s.t

$$\sum_{S' \subseteq S} (M_{S'}(f) - m_{S'}(f)) v(S') < \varepsilon\, v(S) \quad S \in S_2$$

(S' is a refinement of S)



By lemma

$S \leftarrow$ Partition on P

join the endpoints

$\Rightarrow U(f, P') - L(f, P')$ new partition $S' \subseteq S \in S_2$

$= \sum_{S' \subseteq S \in S_2} (M_{S'}(f) - m_{S'}(f)) v(S')$

$+$

$\sum_{S' \subseteq S \in S_1} (M_{S'}(f) - m_{S'}(f)) v(S')$ $\hookrightarrow$ this is done

$< \varepsilon \sum_{S \in S_2} v(S) + 2M\varepsilon$

$\leq \varepsilon\, v(A) + 2M\varepsilon$

$= (v(A) + 2M)\varepsilon$

so $U(f, P') - L(f, P')$ can be made as small as we like by taking P appropriately

$\Rightarrow f$ is integrable

($\Rightarrow$) suppose f is integrable we want to show : B has measure 0

Note $B = B_1 \cup B_{\frac{1}{2}} \cup B_{\frac{1}{3}} \cup \dots\dots$

so, enough to show $B_{\frac{1}{n}}$ has measure 0 for every $n \geq 1$

$B_{\frac{1}{n}} = \left\{ x \,\middle|\, O(f, x) \geq \frac{1}{n} \right\}$, as f is integrable, given $\varepsilon > 0$, $\exists$ Partition P

$U(f, P) - L(f, P) < \frac{\varepsilon}{n}$

$\Rightarrow \sum_{S} (M_S(f) - m_S(f)) v(S) < \frac{\varepsilon}{n}$

$\Rightarrow \sum_{S : S \cap B_{\frac{1}{n}} \neq \phi} (M_S(f) - m_S(f)) v(S) < \frac{\varepsilon}{n}$ (coutable S cover $B_{\frac{1}{n}}$)

$\Rightarrow \frac{1}{n} \sum_{S \cap B_{\frac{1}{n}} \neq \phi} v(S) < \frac{\varepsilon}{n}$ (as $M_S(f) - m_S(f) > \frac{1}{n}$ by defination)

ε'

$\Rightarrow \sum_{S \cap B_{\frac{1}{n}} \neq \phi} v(S) < \varepsilon$

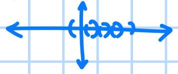
and $\{ S \mid S \cap B_{\frac{1}{n}} \neq \phi \}$ is a cover of $B_{\frac{1}{n}}$ by finitely many closed retagles

$\Rightarrow B_{\frac{1}{n}}$ has measure / content 0 (as $B_{\frac{1}{n}}$ is closed)

$\Rightarrow B = B_1 \cup B_{\frac{1}{2}} \cup \dots$ has measure 0

17th Feb :

Exe : prove $\mathbb{R} \subseteq \mathbb{R}^2$ has measure 0



length = 2

(general $\mathbb{R}^{n-1} \subseteq \mathbb{R}^n$ has measure = 0)

$\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (n, n+2)$

now let $u_i = [n, n+2] \times [\varepsilon']$ → Here take care of indexing

$[i, i+2](\varepsilon')$ for $i > 0$

and $[i, i+2]\, \varepsilon''$ for $i < 0$

$v(u_i) = 2\varepsilon'$

then $\sum_{i=1}^{\infty} v(u_i) < 2 \sum_{i=1}^{\infty} \varepsilon_i'$

for $\varepsilon_1' = \frac{1}{2}\left(\frac{\varepsilon}{2}\right)$

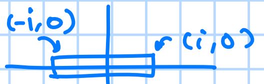
$\varepsilon_2' = \frac{1}{2}\left(\frac{\varepsilon}{2^2}\right)$

$\vdots$

$\sum \varepsilon_i' = \frac{1}{2} \times \varepsilon$

so $\sum v(u_i) < \varepsilon$

or



h_i s.t $h_i(2i) < \frac{\varepsilon}{2^i}$

Note : for $\mathbb{R}^2 \subseteq \mathbb{R}^3$ measure is 0,

rectangle: $[-i, i] \times [-i, i] \times [h_i]$

$v(u_i) = (2i)^2 h_i < \varepsilon/2^i$

$\Rightarrow h_i < \frac{\varepsilon}{(2^i)(2i)^2}$

so $v(u_i) < \frac{\varepsilon}{2^i}$

$\Rightarrow \sum v(u_i) < \varepsilon$

Note : $[-i, i] \times \{0\}$ this is also a closed rectangle

Theorm : let A be a closed rectangle and let $f : A \to \mathbb{R}$ be a bounded function.
$B = \{x \mid f$ is not cont at $x\}$ then :

f is integrable

$\Updownarrow$

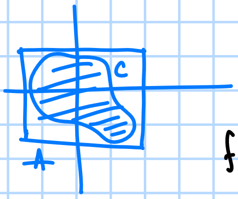
B is a set of measure 0

(this theorm B has measure 0 ⇔ f is int proved and will be needed)

so far we have defined integrals of a function on retangles A.
what about integrating on other sets C?

for $C \subseteq \mathbb{R}^n$, we can define the charatistic funtion χ_C as follows:

$$\chi_C(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}$$



If $C \subseteq A$ for some closed rectangle A, and $f : C \to \mathbb{R}$, then define

$$\int_C f := \int_A f \circ \chi_C$$

provided that $f \circ \chi_C$ is integrable

(we have to check $f \circ \chi_C$ int) (if f and χ_C are integrable $\Rightarrow f \circ \chi_C$ is integrable)

Note: $f_1, f_2 : A \to \mathbb{R}$ are integrable, then $f_1 f_2$ is also integrable

Theorm: The function $\chi_C : A \to \mathbb{R}$ is integrable iff boundary (C) has measure 0.

proof: From theorm of last week,
χ_C is integrable iff $\underbrace{\text{the set of points at which } \chi_C \text{ is not cont}}_{B}$ has measure 0.

we want to show, B = Boundary of C

If $x \in \text{int}(C)$ then by defination,
$\exists$ open retagles U s.t
$x \in U$
and $U \subseteq C$
$\Rightarrow \chi_C \equiv 1$ on U

$\Rightarrow \chi_C$ is cont at x —— ①

if $x \in \text{ext}(C)$ then by defination
$\exists$ open rectangle V s.t
$x \in V$
and $V \subseteq A \setminus C$
$\Rightarrow \chi_C \equiv 0$ on V

$\Rightarrow \chi_C$ is cont at x —— ②

Now if $x \in$ Boundary of C then $\forall$ open retangle W
st
$x \in W$
$\Rightarrow$ W contain points $y \in C$
as well as $z \in A \setminus C$
$\Rightarrow$ any open retangle containg x
has points where $\chi_C = 1$
as well as points $\chi_C = 0$

$\Rightarrow \chi_C$ is not cont at x —— ③

now,
$x \in \text{Boundry}(C) \Rightarrow x \in B$ from ③
$x \notin \text{Boundry}(C) \Rightarrow x \notin B$ from ① and ②

$\Rightarrow x \in \text{Boundry}(C) \Leftrightarrow x \in B$
$\Rightarrow \text{Boundry}(C) = B$

($x \in B \Rightarrow x \in$ Boundary of C
$x \in$ Boundry $C \Rightarrow x \in B$)

(Boundary of C has points which are discont)

Defn : A boundary of set C whose boundary has measure 0 is called **Jordan-measurable**.

Defn : The integral $\int_C 1$ is called the (n-dimensional) content, or the (n-dimm) volume of C.

Evaluating integrals:

1-variable, we have FTC s.t

if f is cts on [a,b], and if g is a function s.t

$$g'(x) = f(x), \forall x \in [a,b] \text{ then}$$

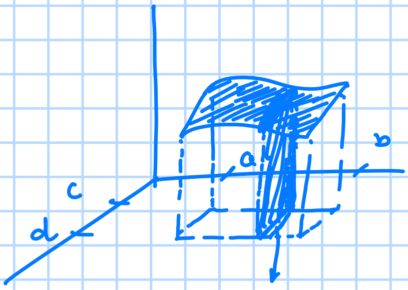
$$\int_a^b f = g(b) - g(a)$$

for more than 1 variable, for evaluating $\int_A f$, i.e $\int_{[a_1,b_1]\times\cdots\times[a_n,b_n]} f$

try to reduce the problem to computing n 1-variable integrals.

Eg: $f : \mathbb{R}^2 \to \mathbb{R}$ is cont

$$\int_{[a,b]\times[c,d]} f = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$



$$\int_c^d f(x,y)\,dy = \text{volume of this slice}$$

d this slice

x → const, y → variable

$x-t$ — x — $x+t$

If $g_x(y) = f(x,y)$, area above $\{x\}\times[c,d]$ and below f is

$$\int_c^d g_x = \int_c^d f(x,y)\,dy$$

vol above $[t_{i-1}, t_i]\times[c,d]$ and below f is:

$$(t_i - t_{i-1}) \times \int_c^d f(x,y)\,dy$$

for $x \in [t_{i-1}, t_i]$

$$\Rightarrow \int_{[a,b]\times[c,d]} f = \sum_{i=1}^n \int_{[t_{i-1},t_i]\times[c,d]} f$$

$$\approx \sum_{i=1}^n (t_i - t_{i-1}) \int_c^d f(x,y)\,dy$$

(for $x_i \in [t_{i-1}, t_i]$)

Note : we can guess that if f above is integrable $h(x) = \int_c^d f(x,y)\,dy$ is integrable

(guess can be solved using fubinis theorm) and $\int_a^b h(x)\,dx = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_A f$

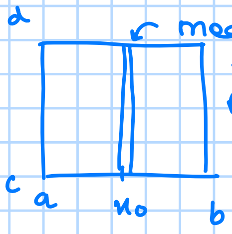
18th Feb :

Recap : guess was that if $f : [a,b] \times [c,d] \longrightarrow \mathbb{R}$ then if f is integrable

$$h(x) = \int_c^d f(x,y)\,dy \text{ is integrable and}$$

$$\int_a^b h(x)\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_{[a,b]\times[c,d]} f$$

(this is an assumption as $h(x) = \int_c^d f(x,y)\,dy$ cannot be true)



measure 0 then not integrable at x_0

but $\int_c^d f(x_0, y)\,dy$ does not make sense

$\therefore$ this $h(x) = \int_c^d f(x,y)\,dy$ was conditional

Def^n : If $f : A \longrightarrow \mathbb{R}$ is a bounded function, the lower integral of f is

$$L\int_A f = \sup_P L(f,P)$$

($\sup L(f,P)$ = Lower sum maximum)

upper integral of f is :

$$U\int_A f = \inf_P U(f,P)$$

Note : even if f is not integrable, as it is bounded, we would have $L\int_A f$ and $U\int_A f$, moreover if f is integrable then

$$L\int_A f = U\int_A f \qquad \left(\sup_P L(f,P) = \inf_P U(f,P) \right)$$

Note : $L\int_A f = \underline{\int_A} f$ is munkre's notation

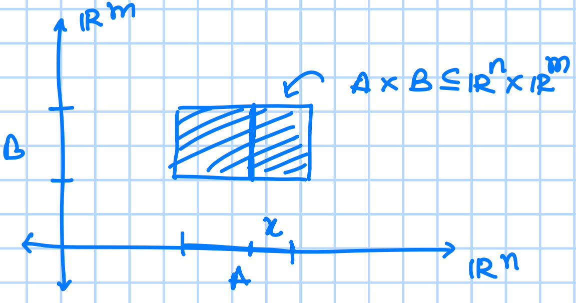
$$U\int_A f = \overline{\int_A} f$$

Theorm : (Fubini's theorm) let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$ be closed rectangles and let $f : A \times B \longrightarrow \mathbb{R}$ be integrable. For $x \in A$, let g_x be defined by

$$g_x(y) = f(x,y) \text{ (const wrt } x\text{)}$$

let, $\mathcal{L}(x) = L\int_B g_x = L\int_B f(x,y)\,dy$

$$\mathcal{L}(x) = L\int_B f(x,y)\,dy$$



and $\mathcal{U}(x) = U\int_B g_x = U\int_B f(x,y)\,dy$

then $\mathcal{L}(x)$, $\mathcal{U}(x)$ are integrable on A and :

$$\int_{A\times B} f = \int_A \mathcal{L} = \int_A L\int_B f(x,y)\,dy\,dx$$

$$\int_{A\times B} f = \int_A \mathcal{U} = \int_A U\int_B f(x,y)\,dy\,dx$$

Remarks :

① under the assumption of the theorem, we also have

$$\int_{A\times B} f = \int_B L\left(\int_A f(x,y)\,dx\right)dy$$
$$= \int_B U\left(\int_A f(x,y)\,dx\right)dy$$

② If each g_x is integrable (if f is cont $\Rightarrow$ g_x is integrable)

$$\int_{A\times B} f = \int_A \int_B f(x,y)\,dy\,dx$$

$\left(\mathcal{L}(x) = \mathcal{U}(x) \Rightarrow \int_A \mathcal{L}\int_B f(x,y) = \int_A \mathcal{U}\int_B f(x,y) \Rightarrow \int_A\int_B = \int_{A\times B}\right)$

③ If $A = [a_1, b_1] \times \dots \times [a_n, b_n]$
and if f is sufficiently nice, we can use fubini's theorem repeatedly to obtain :
(sufficiently nice f required)

$$\int_A f = \int_{a_n}^{b_n}\left(\int_{a_{n-1}}^{b_{n-1}} \cdots \left[\int_{a_1}^{b_1} f(x^1, x^2, \dots x^n)\,dx^1\right]dx^2\right) \dots dx^n$$

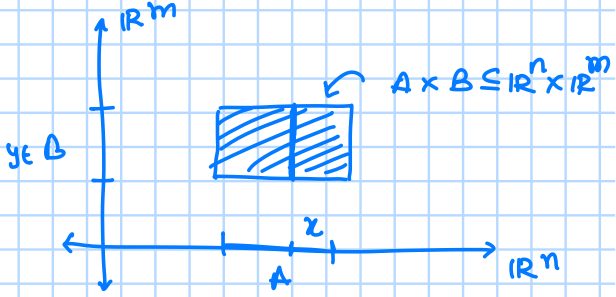
20th Feb :

Theorm: (Fubini's theorm) let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$ be closed rectangles and let $f : A \times B \longrightarrow \mathbb{R}$ be integrable. For $x \in A$, let g_x be defined by

$$g_x(y) = f(x,y)$$

let, $\mathcal{L}(x) = L\int_B g_x = L\int_B f(x,y)dy$

$$\mathcal{L}(x) = L\int_B f(x,y)dy$$



and $\quad \mathcal{U}(x) = U\int_B g_x = U\int_B f(x,y)dy$

then $\mathcal{L}(x)$, $\mathcal{U}(x)$ are integrable on A and:

$$\int_{A\times B} f = \int_A \mathcal{L} = \int_A L\int_B f(x,y)dy\,dx$$

$$\int_{A\times B} f = \int_A \mathcal{U} = \int_A U\int_B f(x,y)dy\,dx$$

proof:

let P_A be a partition for A, let P_B be a partition for B. Then

$$P = (P_A, P_B)$$

is a partition for $A\times B$ s.t every subretagle of P is of form $S_A \times S_B$ for S_A a subrectagle of P_A and S_B a subrectagle of P_B.

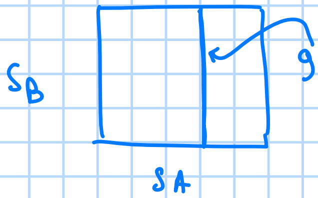
then $L(f,P) = \sum_S m_S(f)\, v(S)$

$$= \sum_{S_A, S_B} m_{S_A\times S_B}(f)\, v(S_A \times S_B)$$

$$= \sum_{S_A}\Big(\sum_{S_B} m_{S_A\times S_B}(f)\, v(S_B)\Big) v(S_A)$$ ↙ over entire strip

if $x \in S_A$, then $m_{S_A\times S_B}(f) \leq m_{S_B}(g_x)$ ↘ inf over li

↪ infimum over whole



↙ const
$g_x = f(x,y)$

$$\sum_{S_B} m_{S_A\times S_B}(f) \times v(S_B) \leq \sum_{S_B} m_{S_B}(g_x)\, v(S_B)$$

$$\leq L\int_B g_x = \mathcal{L}(x) \quad \forall x \text{ in } S_A$$

↓ as this is defined as supremum of all x

this is true for each $x \in S_A$ so, ↙ inf over all $x \in S_A$

$$\sum_{S_B} m_{S_A\times S_B}(f)\, v(S_B) \leq m_{S_A}(\mathcal{L})$$

$$L(f,P) = \sum_{S_A}\Big(\sum_{S_B} m_{S_A\times S_B}(f)\, v(S_B)\Big) v(S_A) \leq \sum_{S_A} m_{S_A}(\mathcal{L})\, v(S_A) = L(\mathcal{L}, P_A)$$

similarly $U(\upsilon, P_A) \leq U(f,P)$

$$\Rightarrow L(f,P) \leq \underbrace{L(\mathcal{L}, P_A) \leq U(\mathcal{L}, P_A)}_{\downarrow} \leq U(\upsilon, P_A) \leq U(f,P)$$

as $\mathcal{L} \leq \upsilon$ $(\forall x \in A)$

this is as $\forall P_A$ $L(g', P_A) \leq U(g', P_A)$

as f is integrable, $\sup_P L(f,P) = \inf_P U(f,P)$

$$= \int_{A \times B} f$$

$$\Rightarrow \sup_{P_A} L(\mathcal{L}, P_A) = \inf_{P_A} U(\mathcal{L}, P_A)$$

$$= \int_{A \times B} f$$

$\Rightarrow \mathcal{L} : A \to \mathbb{R}$ is integrable and

$$\int_A \mathcal{L} = \int_{A \times B} f$$

$$\int_A \mathcal{L} = \int_A L \int_B f(x,y)\, dy\, dx$$

similarly $L(f,P) \leq L(\mathcal{L}, P_A) \leq L(\upsilon, P_A) \leq U(\upsilon, P_A) \leq U(f,P)$

we can see that

$\upsilon : A \to \mathbb{R}$

is integrable, and

$$\int_A \upsilon = \int_{A \times B} f$$

where

$$\int_A \upsilon = \int_A U \int_B f(x,y)\, dy\, dx$$

Note : If f is continious, then $\int_{A \times B} f(x,y) = \int_A \int_B f(x,y)\, dy\, dx$

$$= \int_B \int_A f(x,y)\, dx\, dy$$

Theorm : let $f : \mathbb{R}^n \to \mathbb{R}$, if $D_i f, D_j f, D_{i,j}, D_{j,i} f$ are continious in an open set containg a, then

$D_{i,j} f(a) = D_{j,i} f(a)$ (we need $D_i f$, $D_j f$ also cont, stronger condition)

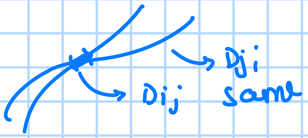
proof :

define $g(x,y) = f(a^1, \ldots, \underset{i^{th}}{x}, \ldots \underset{j^{th}}{y}, \ldots a^n)$

write $P_0 = (a^i, a^j)$

then to show :

$D_{1,2}\, g(P_0) = D_{2,1}\, g(P_0)$



let's use contradiction,

if not true then wlog:

$D_{1,2}\,g(P_0) > D_{2,1}\,g(P_0)$ (wlog case)

$\Rightarrow D_{1,2}\,g(P_0) - D_{2,1}\,g(P_0) > 0$

as $D_{1,2}\,g(P_0)$, $D_{2,1}\,g(P_0)$ cont on open set cont P_0

$\exists$ closed retagle $A = [a,b]\times[c,d]$ contaning P_0 s.t

$D_{1,2}\,g(P) - D_{2,1}\,g(P) > 0$ $\forall P \in A$ (By defination of continuity)

$$\Rightarrow \int_A (D_{1,2}\,g - D_{2,1}\,g) > 0$$

now $\int_A D_{1,2}\,g = \int_a^b \int_c^d D_{1,2}\,g(x,y)\,dy\,dx$ ($\because$ fubini's theorm and $D_{1,2}$ is cont)

$$= \int_a^b \int_c^d D_2(D_1 g)(x,y)\,dy\,dx$$

$$= \int_a^b \left(D_1 g(x,d) - D_1\,g(x,c)\right)dx$$

$$= g(b,d) - g(a,d) - g(b,c) + g(a,c)$$

similarly,

$$\int D_{2,1}\,g = g(b,d) - g(a,d) - g(b,c) + g(a,c)$$

$\Rightarrow \int_A (D_{1,2}g - D_{2,1}\,g) = 0$ $\divideontimes$ ($\because \int_{A\times B} D_{1,2}\,g = \int_{A\times B} D_{2,1}\,g$)

3rd march:

Note: clarification from last time: all D_if, D_jf, $D_{i,j}f$, $D_{j,i}f$ should be continious (find in the theorm in last class).

$$D_if \text{ is continious} \Rightarrow D_i(x^k) \text{ is cont } \forall k$$

$$D_{i,j}f \text{ is cont} \Rightarrow D_{i,j} \text{ is continious}$$

properties of integral:

lemma: let $A \subseteq \mathbb{R}^n$ be a closed rectangle, let $C \subseteq A$. let f, g be funtion from $f, g : C \longrightarrow \mathbb{R}$

let $F, G : C \longrightarrow \mathbb{R}$ be defined by

$F(x) = \max\{f(x), g(x)\}$

$G(x) = \min\{f(x), g(x)\}$

(a) If f, g are continious at x_0, so are F, G

(b) If f, g are integrable over C, then so are F, G.

proof:

(a) suppose f, g are continious at x_0

case I:

$f(x_0) = g(x_0) = r$

$(\therefore F(x_0) = G(x_0) = r)$

given $\varepsilon > 0$, $\exists \delta > 0$ s.t

$|f(x) - r| < \varepsilon \ \forall |x - x_0| < \delta$

$|g(x) - r| < \varepsilon \ \forall |x - x_0| < \delta$

$\Rightarrow |F(x) - r| < \varepsilon$

$|G(x) - r| < \varepsilon$

$\forall |x - x_0| < \delta$

Case II:

wlog $f(x_0) > g(x_0)$, then

$\exists$ open nbd U s.t

$\forall x \in U, f(x) > g(x)$

$\therefore$ on U, $F(x) = f(x)$, $G(x) = g(x)$ $(\forall x \in U)$

$\therefore$ continious at x_0.

(b) suppose f, g are integrable over C

$f\chi_C, g\chi_C$ are integrable over A. (just from defination)

$(B_1, B_2 \subseteq A)$

$\Rightarrow f\chi_C : A \longrightarrow \mathbb{R}$ is continious outside a set B_1 of measure 0.

& $g\chi_C : A \longrightarrow \mathbb{R}$ is continious outside a set B_2 of measure 0.

now, $F\chi_C = \max\{f\chi_C, g\chi_C\}$

$G\chi_C = \min\{f\chi_C, g\chi_C\}$

so, $F\chi_C, G\chi_C$ are continious outside $B_1 \cup B_2$ and $B_1 \cup B_2$ has measure 0.

Also, $F\chi_C, G\chi_C$ are bounded ($\because$ $f\chi_C, g\chi_C$ are bounded)

$\Rightarrow F\chi_C, G\chi_C$ are integrable over A

$\Rightarrow F, G$ are integrable over C.

Theorm: (Properties of the integral) let $A \subseteq \mathbb{R}^n$, be closed rectagle. let $C \subseteq A$. f, g are bounded funtions.

(a) (linearity) If f, g are integrable over C, then $a.f + b.g$ is also integrable and $\int_C (af + bg) = a\int_C f + b\int_C g$

(b) (comparison) Suppose f, g are integrable over C. If $f(x) \leq g(x)$ $\forall x \in C$

$$\Rightarrow \int_C f \leq \int_C g$$

also, $|f|$ is integrable

and $\left|\int_C f\right| \leq \int_C |f|$

(c) (Monotonicity) let $T \subseteq C$, if f is non-negative on C, and integrable over T and C $\Rightarrow$

$$\int_T f \leq \int_C f$$

(d) (Additivity) If $C = C_1 \cup C_2$ and f is integrable over C_1 and C_2 then f is integrable on C_1, $C_1 \cap C_2$ and

$$\int_{C_1 \cup C_2} f = \int_{C_1} f + \int_{C_2} f - \int_{C_1 \cap C_2} f$$

proof:

(a) $(af + bg)\chi_C = af\chi_C + bg\chi_C$

so wlog, enough to prove the statement for the integrals of function defined on A

so, wlog $f, g : A \to \mathbb{R}$ s.t f, g are integrable over A

now $\Rightarrow$ f, g are cont outside B_1, B_2 (sets of measure 0)

$\therefore$ $af + bg$ is continuous outside of $B_1 \cup B_2$ which is a set of measure 0

$\Rightarrow$ $af + bg$ is integrable over A

now

$$\int_A (af + bg) = a\int_A f + b\int_A g$$

$F(x) = \int f(x)$

$\Rightarrow F'(x) = f(x)$

$\Rightarrow (kF(x))' = kF'(x) = kf(x)$

$\Rightarrow kF(x) = \int kf(x) = k\int f(x)$

$\int f + g$ $\quad F'(x) = f(x)$

$G'(x) = g(x)$

$(F + G)' = F' + G' = f + g$

$(F + G)' = f + g$

$\Rightarrow F + G = \int f + g = \int f + \int g$

(b) Enough to prove this for integral over A

$(\because f\chi_C \leq g\chi_C)$

so wlog, assume that $f(x) \leq g(x)$ $\forall x \in A$

$f, g : A \to \mathbb{R}$

If S is any rectangle contained in A, then

$$m_S(f) \leq f(x) \leq g(x) \quad \forall x \in S$$

$$\Rightarrow m_S(f) \leq m_S(g) \quad \forall x \in S$$

if P is any partition of A,

$$L(f, P) \leq L(g, P) \leq \int_A g$$

as P is arbitrary:

$$\int_A f \leq \int_A g$$

note that $|f| : A \to \mathbb{R}$ is

$|f| = \max\{f, -f\}$ i.e

$|f(x)| = \max\{f(x), -f(x)\}$ now

by lemma, $f, -f$ integrable

$\Rightarrow |f|$ is integrable over A

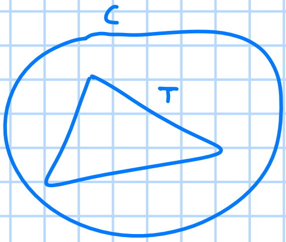
also, $-|f(x)| \leq f(x) \leq |f(x)|$

$$\Rightarrow -\int_A |f| \leqslant \int_A f \leqslant \int_A |f|$$

$$\Rightarrow \left|\int_A f\right| \leqslant \int_A |f|$$

(c) $(T \subseteq C)$ (monotonicity)

If $f \geqslant 0$ and $T \subseteq C$ then
$f\chi_T \leqslant f\chi_C$



Apply the comparison property (property (b))

$$\Rightarrow \int_A f\chi_T \leqslant \int_A f\chi_C$$

$$\Rightarrow \int_T f \leqslant \int_C f$$

(d) let $T = C_1 \cap C_2$

<u>Case I</u>: $f \geqslant 0$ on $C = C_1 \cup C_2$

By assumption
$f\chi_{C_1}$, $f\chi_C$ are both integrable over A

then, $(f\chi_C)(x) = \max\{(f\chi_{C_1})(x), (f_1\chi_{C_2})(x)\}$

($f\chi_C, f\chi_T$ only when $f \geqslant 0$) $(f\chi_T)(x) = \min\{(f\chi_{C_1})(x), (f_1\chi_{C_2})(x)\}$

(lemma) $\Rightarrow f\chi_C$ and $f\chi_T$ are integrable over A.

<u>Case II</u>: (general case)

let $f_+(x) = \max\{f(x), 0\} \Rightarrow f_+$ is integrable

$f_-(x) = \max\{-f(x), 0\} \Rightarrow f_-$ is integrable

then f_+, f_- are both non, negative functions
$f_+, f_- \geqslant 0$

f_+, f_- are integrable over C and T.

(Apply lemma and case I)

now $f = f_+ - f_-$ and linearity implies f is integrable over C and T.

$$f\chi_C = f\chi_{C_1} + f\chi_{C_2} - f\chi_{C_1 \cap C_2}$$ (from the fact that f is integrable and inclusion-exclusion principle)

$$\Rightarrow \int_C f = \int_{C_1} f + \int_{C_2} f - \int_{C_1 \cap C_2} f$$